Introduction to Nonlinear Control

Stability, control design, and estimation

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Part I:

Chapter 1: Nonlinear Systems - Fundamentals

Australian National University

Nonlinear Systems - Fundamentals

Section 1

State Space Models

State Space Models

(Time-invariant) First order differential equations (or autonomous system):

$$\dot{x}(t) = \frac{d}{dt}x(t) = f(x(t)), \quad f : \mathbb{R}^n \to \mathbb{R}^n \tag{1}$$

A *solution* of (1) is an absolutely continuous function that satisfies (1) for almost all t.

Non-autonomous/time-varying system:

$$\dot{x}(t) = f(t, x(t)), \qquad f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n \tag{2}$$

Theorem (Existence & Uniqueness)

Given $x_0 \in \mathbb{R}^n$, $r > 0$, and $0 \leq t_0 < t_1$, let $f(t,x)$ be piecewise continuous in t and satisfy the (local) Lipschitz condition

$$|f(t,x) - f(t,y)| \leq L|x - y|$$

for an $L > 0$, for all $x, y \in \{\xi \in \mathbb{R}^n : |\xi - x_0| \leq r\}$ and $t \in [t_0, t_1]$. Then there exists $\delta > 0$ so that

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0$$

has a unique solution over $[t_0, t_0 + \delta]$.

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From n^{th}-order to first order:

$$\frac{d^n}{dt^n}y(t) = \phi\left(y(t), \dot{y}(t), \ldots, \frac{d^{n-1}}{dt^{n-1}}y(t)\right), \quad \phi : \mathbb{R}^n \to \mathbb{R}$$

Define

$$x_1 = y, \quad x_2 = \dot{y}, \quad x_3 = \ddot{y}, \ \ldots\ , \ x_n = \frac{d^{n-1}}{dt^{n-1}}y, \qquad (3)$$

Corresponding first order system:

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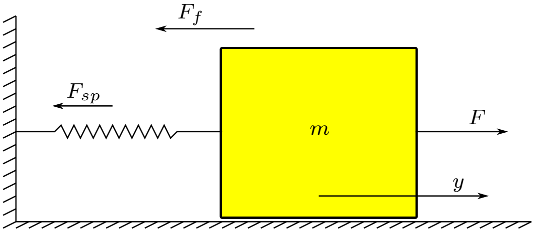
Systems with external inputs $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$:

$$\dot{x} = f(x,u), \qquad \dot{x} = f(x,w),$$

- $u : \mathbb{R}^n \to \mathbb{R}^m$, $x \mapsto u(x)$ ← degree of freedom
- $w : \mathbb{R} \to \mathbb{R}^m$, $t \mapsto w(t)$ ← exogenous signal (disturbance or reference)

State Space Models (Example: Mass-Spring System)



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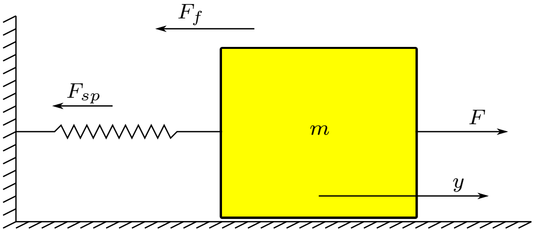

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Newton's second law of motion:

$$m\ddot{y} = F - F_f - F_{sp} = F - c\dot{y} - ky \qquad (4)$$

($F_f = c\dot{y}$ is viscous friction & $F_{sp} = ky$ is a linear spring)

From second order to first order: ($x_1 = y$, $x_2 = \dot{y}$, $u = F$)

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -\tfrac{k}{m}x_1(t) - \tfrac{c}{m}x_2(t) + \tfrac{1}{m}u(t) \end{aligned}$$

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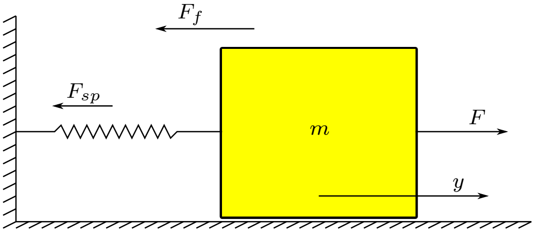

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Potential energy (for $u = 0$):

$$\tfrac{1}{2}ky^2 = \tfrac{1}{2}kx_1^2$$

Kinetic energy (for $u = 0$):

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}m(\dot{y})^2 = \tfrac{1}{2}mx_2^2, \qquad (v : \text{velocity of the block})$$

$\rightsquigarrow$ How does the total energy of the mass-spring system evolve with time?

Total energy: potential + kinetic energy

$$E(x_1, x_2) = E(x) = \tfrac{1}{2}kx_1^2 + \tfrac{1}{2}mx_2^2 \geq 0.$$

The time derivative of E:

$$\begin{aligned}\tfrac{d}{dt}E(x(t)) &= \tfrac{d}{dt}\left(\tfrac{1}{2}kx_1(t)^2 + \tfrac{1}{2}mx_2(t)^2\right)\\ &= kx_1\dot{x}_1(t) + mx_2\dot{x}_2(t)\\ &= kx_1(t)x_2(t) - kx_1(t)x_2(t) - cx_2(t)^2 = -cx_2(t)^2 \leq 0\end{aligned}$$

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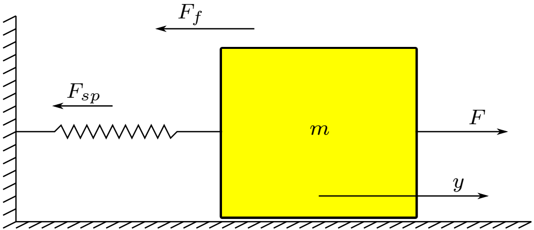

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E is positive, $\dot{E}$ is decreasing ⇝ eventually, the block must stop moving.

⇝ Where is the block going to stop?

State Space Models (Equilibria and pairs of induced equilibria)

Definition (Equilibrium, $\dot{x} = 0$)

The point $x^e \in \mathbb{R}^n$ is called an equilibrium of the system $\dot{x} = f(x)$ or $\dot{x} = f(t, x)$, respectively, if

$$\begin{aligned} \tfrac{d}{dt}x(t) &= f(x^e) = 0, \\ \tfrac{d}{dt}x(t) &= f(t, x^e) = 0 \qquad \forall t \in \mathbb{R}_{\geq 0}. \end{aligned}$$

The pair $(x^e, u^e) \in \mathbb{R}^n \times \mathbb{R}^m$ is called an equilibrium pair of the system $\dot{x} = f(x, u)$ if

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- Without loss of generality $x^e = 0$ (or $(x^e, u^e) = 0$).
- To see this, consider coordinate transf. $z = x - x^e$.
- Then

$$\tfrac{d}{dt}z(t) = \tfrac{d}{dt}x(t) - \tfrac{d}{dt}x^e = f(x(t)) = f(z(t) + x^e).$$

and

$$\hat{f}(z) \doteq f(z + x^e) \qquad \text{yields} \qquad \dot{z} = \hat{f}(z)$$

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- Use a similar translation $z = x - x^e$ and $v = u - u^e$ to shift an equilibrium pair to the origin.

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In the case that $u = 0$:

- The first equation implies that $x_2 = 0$.
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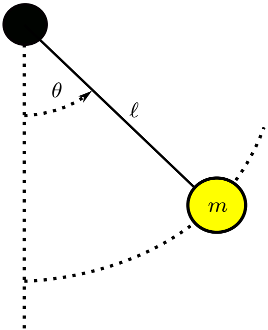
Exercise: How do the equilibrium pairs look like?

State Space Models (Example: Pendulum)



Equilibria:

$$0 = x_2, \qquad 0 = -\tfrac{g}{\ell}\sin x_1 - \tfrac{k}{m}x_2$$

$\rightsquigarrow x^e = [n\pi, 0]^T$ for $n = 0, \pm 1, \pm 2$, etc..

System dynamics (by balancing forces):

$m\ell\ddot{\theta} = -mg\sin\theta - k\ell\dot{\theta}, \qquad k > 0$ friction coefficient

With $x_1 = \theta$ and $x_2 = \dot{\theta}$:

$$\dot{x}_1(t) = x_2(t),$$
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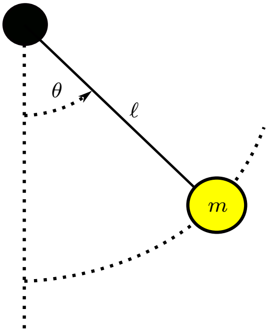

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Potential energy: $mg\ell(1-\cos\theta) = mg\ell(1-\cos x_1)$

Kinetic energy: $\frac{1}{2}mv^2 = \frac{1}{2}m\ell^2\dot{\theta}^2 = \frac{1}{2}m\ell^2x_2^2$.

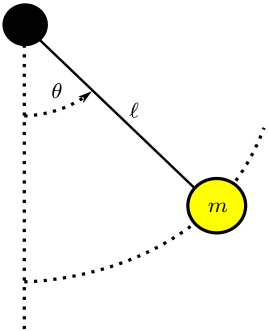
Time evolution of the total energy:

$$\begin{aligned}\tfrac{d}{dt}E(x(t)) &= (mg\ell\sin x_1)\,\dot{x}_1 + m\ell^2x_2\dot{x}_2\\ &= (mg\ell\sin x_1)\,x_2 - m\ell^2x_2\left(\tfrac{g}{\ell}\sin x_1\right) - m\ell^2x_2\left(\tfrac{k}{m}x_2\right)\\ &= -k\ell^2x_2(t)^2.\end{aligned}$$

Note that:

- If $\dot{\theta} = x_2 \neq 0$ then $E(x(t))$ is decreasing.
- However, since the pendulum exhibits multiple equilibria, it is not clear where the equilibrium is going to stop.
- (We will return to this example)

State Space Models (Systems with outputs)



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Note that:

- Full knowledge of $x \in \mathbb{R}^n$ is usually not available.
- For the pendulum a common implementation includes a sensor for measuring the angle $\theta = x_1$ but no velocity sensor.

$\rightsquigarrow$ We can measure $\theta = x_1$ but not $\dot{\theta} = x_2$.

- System with output:

$$\dot{x} = f(x, u)$$
$$y = h(x, u)$$

- In the example of the pendulum:

$$y = h(x) = x_1$$

State Space Models: Notational Convention

A common abuse of notation: (we drop the t-argument)

$$\dot{x} = f(x), \quad x_0 \in \mathbb{R}^n$$

Absolutely continuous solutions $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ such that $x(0) = x_0 \in \mathbb{R}^n$ satisfy

$$\tfrac{d}{dt}x(t) = f(x(t)) \quad \text{for almost all } t \in \mathbb{R}_{\geq 0}$$

Thus, depending on the context

- $x \in \mathbb{R}^n$ denotes a vector
- $x(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ denotes a function

The time-derivative of energy-like functions $E : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$:

$$\tfrac{d}{dt}E(x(t)) = \langle \nabla E(x), \dot{x} \rangle = \langle \nabla E(x), f(x) \rangle$$

where (gradient)

$$\nabla E(x) = \begin{bmatrix} \frac{\partial}{\partial x_1} E(x) \\ \frac{\partial}{\partial x_2} E(x) \\ \vdots \\ \frac{\partial}{\partial x_n} E(x) \end{bmatrix}.$$

State Space Models: Notational Convention

A common abuse of notation: (we drop the t-argument)

$$\dot{x} = f(x), \quad x_0 \in \mathbb{R}^n$$

Absolutely continuous solutions $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ such that $x(0) = x_0 \in \mathbb{R}^n$ satisfy

$$\tfrac{d}{dt}x(t) = f(x(t)) \quad \text{for almost all } t \in \mathbb{R}_{\geq 0}$$

Thus, depending on the context

- $x \in \mathbb{R}^n$ denotes a vector
- $x(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ denotes a function

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In the example of the mass-spring system (with $u = 0$):

$$\dot{x} = f(x) = \begin{bmatrix} x_2 \\ -\frac{k}{m}x_1 - \frac{c}{m}x_2 \end{bmatrix}$$

$$E(x) = \tfrac{1}{2}kx_1^2 + \tfrac{1}{2}mx_2^2$$

$$\begin{aligned} \tfrac{d}{dt}E(x(t)) &= \langle \nabla E(x), f(x) \rangle \\ &= \begin{bmatrix} kx_1 \\ mx_2 \end{bmatrix}^T \begin{bmatrix} x_2 \\ \frac{1}{m}(-kx_1 - cx_2) \end{bmatrix} \\ &= [kx_1 \;\; mx_2] \begin{bmatrix} x_2 \\ \frac{1}{m}(-kx_1 - cx_2) \end{bmatrix} \\ &= kx_1x_2 + x_2(-kx_1 - cx_2). \end{aligned}$$

State Space Models: (Time) Rescaling

Example: Mass-spring system (with $u = 0$):

$$\dot{x} = f(x) = \begin{bmatrix} x_2 \\ -\frac{k}{m}x_1 - \frac{c}{m}x_2 \end{bmatrix}$$

Assume that $c = 0$ (i.e., no friction)

Time rescaling:

- Let $\omega > 0$ so that $\tau = \omega t$
- Then $\frac{d\tau}{dt} = \omega$, i.e., $d\tau = \omega dt$.
- Let

$$z_1 = x_1 \qquad \text{and}$$

$$z_2 = \frac{d}{d\tau}z_1 = \frac{dt}{d\tau}\frac{d}{dt}z_1 = \frac{dt}{d\tau}\dot{z}_1 = \frac{\dot{x}_1}{\omega} = \frac{x_2}{\omega}$$

- Therefore, we can calculate

$$\frac{d}{d\tau}z_2 = \frac{dt}{d\tau}\dot{z}_2 = \frac{1}{\omega}\frac{\dot{x}_2}{\omega} = -\frac{k}{m\omega^2}x_1 = -\frac{k}{m\omega^2}z_1$$

- Fix $\omega = \sqrt{k/m}$; i.e.,

$$\begin{aligned} \frac{d}{d\tau}z_1 &= z_2 \\ \frac{d}{d\tau}z_2 &= -z_1. \end{aligned}$$

which, qualitatively, captures the behavior of all mass-spring.

State Space Models: (Time) Rescaling

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- Define $\omega = \sqrt{k/m}$ and $\alpha = c\sqrt{m/k}$ then

$$\frac{d}{d\tau}z_1 = z_2,$$
$$\frac{d}{d\tau}z_2 = -z_1 - \alpha z_2.$$

State Space Models: (Time) Rescaling

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$$\frac{d}{d\tau}z_2 = -z_1 - \alpha z_2.$$

Summary:

- Instead of three parameters $m, k, c > 0$, we have an equivalent representation with one parameter $\alpha > 0$.

State Space Models: (State) Rescaling

Example: Pendulum

$$\dot{x} = f(x) = \begin{bmatrix} x_2 \\ -\frac{g}{\ell}\sin x_1 - \frac{k}{m}x_2 \end{bmatrix}$$

Time & state rescaling:

- Let $\omega > 0$ so that $\tau = \omega t$
- Then $\frac{d\tau}{dt} = \omega$, i.e., $d\tau = \omega dt$.
- Let

$$z_1 = \frac{x_1}{\beta} \qquad \text{and}$$

$$z_2 = \frac{d}{d\tau}z_1 = \frac{dt}{d\tau}\dot{z}_1 = \frac{\dot{x}_1}{\omega\beta} = \frac{x_2}{\omega\beta}.$$

- Then

$$\begin{aligned} \frac{d}{d\tau}z_2 = \frac{dt}{d\tau}\dot{z}_2 = \frac{1}{\omega}\frac{\dot{x}_2}{\beta\omega} &= \frac{1}{\beta\omega^2}\left(-\frac{g}{\ell}\sin x_1 - \frac{k}{m}x_2\right) \\ &= \frac{1}{\beta\omega^2}\left(-\frac{g}{\ell}\sin(\beta z_1) - \frac{k\omega\beta}{m}z_2\right) \\ &= -\frac{g}{\ell\beta\omega^2}\sin(\beta z_1) - \frac{k}{m\omega}z_2. \end{aligned}$$

State Space Models: (State) Rescaling

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- If we define $\omega = k/m$, $\beta = 1$ and $\alpha = g/(\ell\omega^2)$ then

$$\frac{d}{d\tau}z_1 = z_2, \qquad \frac{d}{d\tau}z_2 = -\alpha\sin z_1 - z_2.$$

State Space Models: (State) Rescaling

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⇝ The rescaled system makes certain qualitative elements clearer.

- For example, observe that near $x = 0$, $\sin x \approx x$ and thus near the point $z_1 = z_2 = 0$, it holds that

$$\frac{d}{d\tau}z_1 = z_2, \qquad \frac{d}{d\tau}z_2 = -z_1 - \alpha z_2.$$

⇝ Close to the equilibrium $z_1 = z_2 = 0$, the pendulum and the mass-spring system have the same qualitative behavior.

State Space Models: (State) Rescaling

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⇝ Close to the equilibrium $z_1 = z_2 = 0$, the pendulum and the mass-spring system have the same qualitative behavior.

Reminder: While rescaling can make a system easier to deal with numerically or analytically, it is necessary to reverse the transformations to get back to the specific system of interest.

State Space Models: Comparison Functions

Definition (Class-$\mathcal{P}, \mathcal{K}, \mathcal{K}_\infty, \mathcal{L}, \mathcal{KL}$ functions)

- A continuous function $\rho : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to be positive definite ($\rho \in \mathcal{P}^n$) if $\rho(0) = 0$ and $\rho(x) > 0 \; \forall \, x \in \mathbb{R}^n \backslash \{0\}$.
- For $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ we use $\rho \in \mathcal{P}$.
- $\alpha \in \mathcal{P}$ is said to be of class-$\mathcal{K}$ ($\alpha \in \mathcal{K}$) if α strictly increasing.
- $\alpha \in \mathcal{K}$ is said to be of class-$\mathcal{K}_\infty$ ($\alpha \in \mathcal{K}_\infty$) if $\lim_{s \to \infty} \alpha(s) = \infty$.
- A continuous function $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class-$\mathcal{L}$ ($\sigma \in \mathcal{L}$) if σ is strictly decreasing and $\lim_{s \to \infty} \sigma(s) = 0$.
- A continuous function $\beta : \mathbb{R}^2_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class-$\mathcal{KL}$ ($\beta \in \mathcal{KL}$) if for each fixed $t \in \mathbb{R}_{\geq 0}$, $\beta(\cdot, t) \in \mathcal{K}_\infty$ and for each fixed $s \in \mathbb{R}_{\geq 0}$, $\beta(s, \cdot) \in \mathcal{L}$.

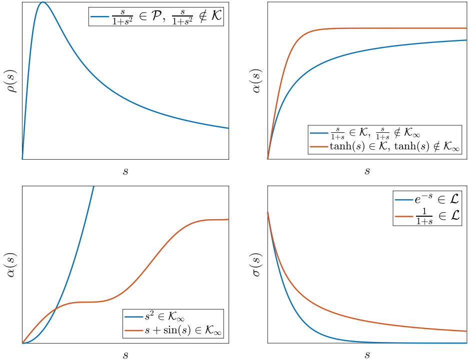
$$\rightsquigarrow \mathcal{K}_\infty \subset \mathcal{K} \subset \mathcal{P}$$

Some properties:

- Class-$\mathcal{K}_\infty$ functions are invertible.
- If $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ then

$$\alpha(s) \doteq \alpha_1(\alpha_2(s)) = \alpha_1 \circ \alpha_2(s) \in \mathcal{K}_\infty.$$

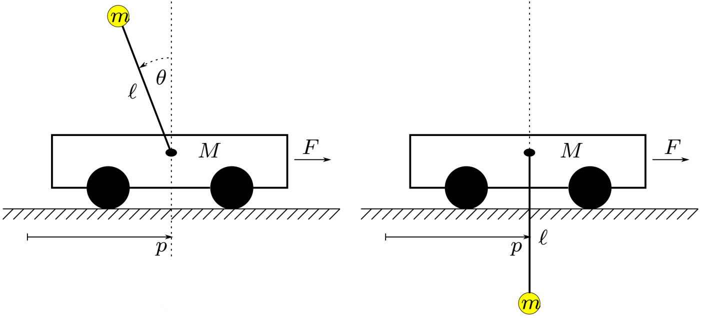
- If $\alpha \in \mathcal{K}$, $\sigma \in \mathcal{L}$ then $\alpha \circ \sigma \in \mathcal{L}$.



Section 2

Examples of Dynamical Systems

Examples of dynamical systems: The inverted pendulum on a cart



General dynamics of a mechanical system:

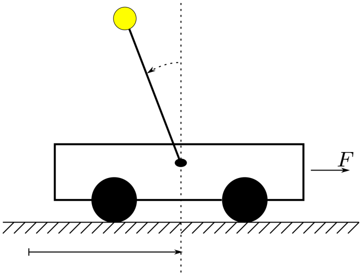
$$M(q)\ddot{q} + C(q,\dot{q}) + K(q) = B(q)u$$

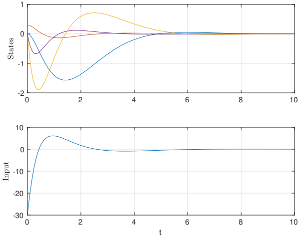
- $M(q)$: inertia matrix
- $C(q,\dot{q})$: Coriolis forces
- $K(q)$: potential energy terms
- $B(q)$: external forces

$$\begin{bmatrix} M+m & -ml\cos(\theta) \\ -ml\cos(\theta) & J+ml^2 \end{bmatrix} \begin{bmatrix} \ddot{p} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} c\dot{p} + ml\sin(\theta)\dot{\theta}^2 \\ \gamma\dot{\theta} - mgl\sin(\theta) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} F$$

$$q = \begin{bmatrix} p \\ \theta \end{bmatrix},$$ parameters, states, inputs

Examples of dynamical systems: The inverted pendulum as a rocket

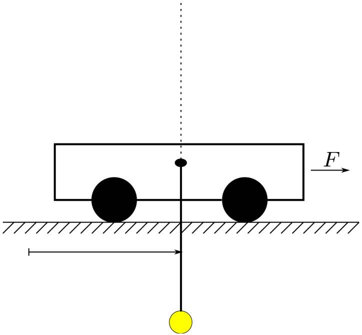


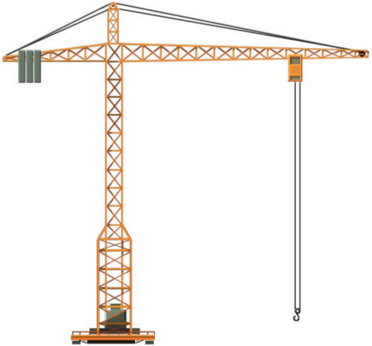


How to define F to *stabilize* the pendulum/rocket in the upright position?

p : position; θ : angle; $\dot{p}$: velocity; $\dot{\theta}$: angular velocity

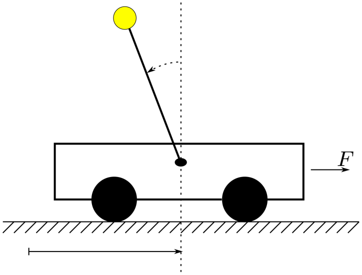
Examples of dynamical systems: The inverted pendulum as a tower crane





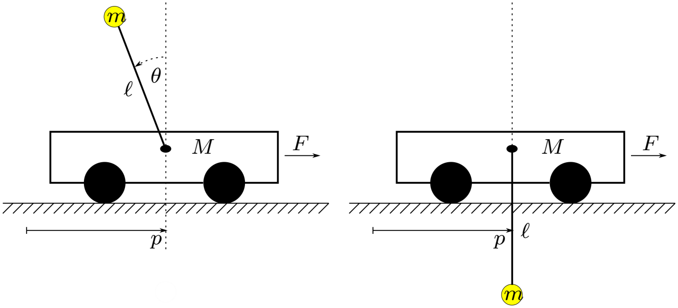
How to define F to *stabilize* the pendulum/tower crane in the lower right position?
How to define F to *drive* the pendulum/tower crane to a specific position?

Examples of dynamical systems: The inverted pendulum as a segway



How to define F to *drive* the pendulum/segway with a fixed velocity? (reference tracking)

Examples of dynamical systems: Local versus global stability properties



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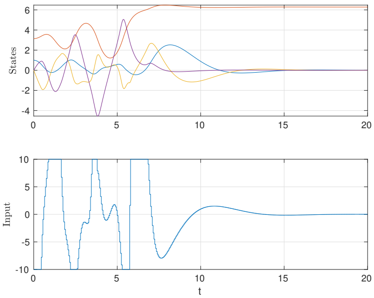

Introduction to Nonlinear Control

Stability, control design, and estimation

Philipp Braun & Christopher M. Kellett
School of Engineering,
Australian National University, Canberra, Australia

Part I:

Chapter 1: Nonlinear Systems - Fundamentals
- 1.1 State Space Models
- 1.2 Examples of dynamical systems

Australian National University