Introduction to Nonlinear Control

Stability, control design, and estimation

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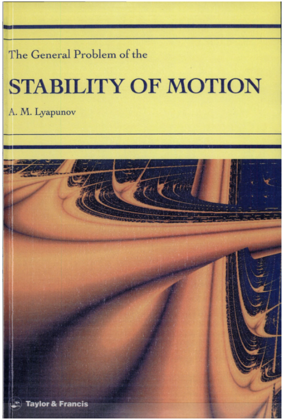
Part I:

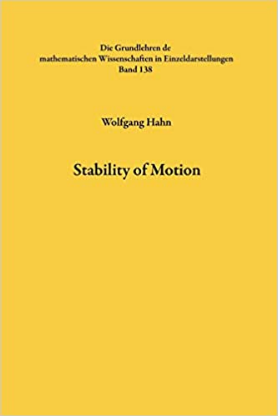
Chapter 2: Nonlinear Systems - Stability Notions

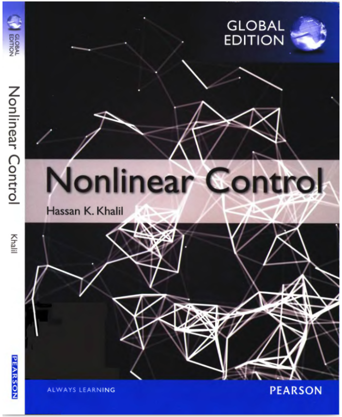
Australian National University

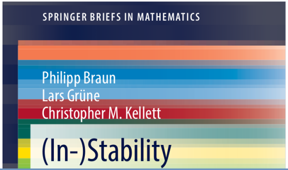
Nonlinear Systems - Fundamentals

Nonlinear Systems - Stability Notions









Section 1

Stability Notions

Stability Notions

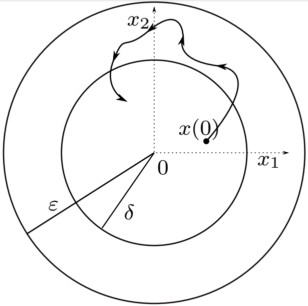
Consider

$$\dot{x} = f(x), \qquad (\text{with } f(x) = 0) \tag{1}$$

Definition (Stability)

The origin is *(Lyapunov) stable* for system (1) if, for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that if $|x(0)| \leq \delta$ then, for all $t \geq 0$,

$$|x(t)| \leq \varepsilon. \tag{2}$$



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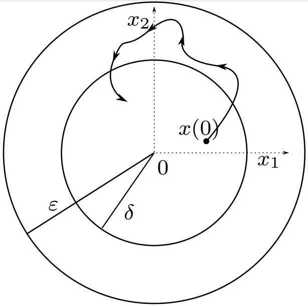

Note that:

- Stability is a property of an equilibrium
- Solutions need to be forward complete

Simple example:

$$\dot{x} = 0, \qquad x(0) = x_0 \in \mathbb{R} \quad \rightsquigarrow \quad x(t) = x_0$$

For any $\varepsilon > 0$, we can choose $\delta = \varepsilon$ so that

$$|x_0| \leq \delta \quad \text{implies} \quad |x(t)| = |x_0| \leq \delta = \varepsilon \quad \rightsquigarrow \quad \text{stability}$$

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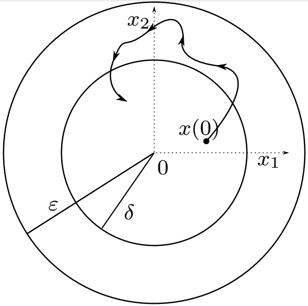

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Equivalent Definition:

The origin is stable if there exists $\alpha \in \mathcal{K}$ and an open neighborhood around the origin $\mathcal{D} \subset \mathbb{R}^n$, such that

$$|x(t)| \leq \alpha(|x(0)|), \qquad \forall\, t \geq 0, \;\; \forall\, x_0 \in \mathcal{D}. \tag{3}$$

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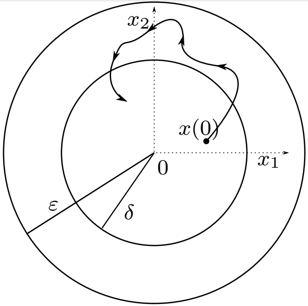

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Definition (Instability)

The origin is *unstable* for system (1) if it is not stable.

Simple Example:

$$\dot{x} = x, \qquad x(0) = x_0 \in \mathbb{R} \quad \rightsquigarrow \quad x(t) = x_0 e^t$$

Stability Notions (Stability/Instability Examples)

Stability Example: (Oscillator)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}$$

Solution:

$$\begin{aligned} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \begin{bmatrix} x_2(0)\sin(t) + x_1(0)\cos(t) \\ -x_1(0)\sin(t) + x_2(0)\cos(t) \end{bmatrix} \\ &= \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \end{aligned}$$

In polar coordinates (r, θ):

$$\begin{aligned} r(t) &= \sqrt{x_1(t)^2 + x_2(t)^2} \\ &= \sqrt{x_1(0)^2 + x_2(0)^2} = |x(0)| = r(0) \\ \theta(t) &= t \end{aligned}$$

For any $\varepsilon > 0$ choose $\delta = \varepsilon$.
Then for any $|x(0)| = r(0) \leq \delta$ we have that

$$|x(t)| = r(t) = r(0) \leq \delta = \varepsilon$$

and so the origin is stable.

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Instability Example: (uncoupled dynamics)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} x_1(0)e^{t} \\ x_2(0)e^{-t} \end{bmatrix}$$

- For initial conditions

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ x_{2,0} \end{bmatrix}, \qquad x_{2,0} \in \mathbb{R}$$

  it holds that $x(t) \to 0$ for $t \to \infty$.
- However, for initial conditions

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} \delta \\ x_{2,0} \end{bmatrix}, \qquad \delta \neq 0, \;\; x_{2,0} \in \mathbb{R}$$

  it holds that $|x(t)| \to \infty$ for $t \to \infty$.
- Thus, the system is unstable

Stability Notions (Attractivity)

Definition (Attractivity)

The origin is *attractive* for $\dot{x} = f(x)$ if there exists $\delta > 0$ such that if $|x(0)| < \delta$ then

$$\lim_{t \to \infty} x(t) = 0. \tag{4}$$

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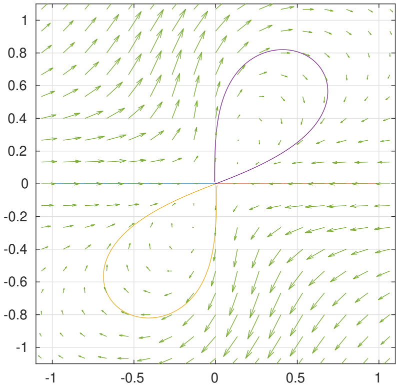
Note that:

- Stability $\not\Rightarrow$ attractivity
  The origin of $\dot{x} = 0$ (with solution $x(t) = x_0$) is stable but not attractive.
- Attractivity $\not\Rightarrow$ stability
  Consider

$$\dot{x}_1 = \frac{x_1^2(x_2 - x_1) + x_2^5}{(x_1^2 + x_2^2)\left(1 + (x_1^2 + x_2^2)^2\right)}$$
$$\dot{x}_2 = \frac{x_2^2(x_2 - 2x_1)}{(x_1^2 + x_2^2)\left(1 + (x_1^2 + x_2^2)^2\right)}.$$



Stability Notions (Asymptotic stability & exponential stability

Definition (Asymptotic stability)

The origin is *asymptotically stable* for $\dot{x} = f(x)$ if it is both stable and attractive.

Definition ($\mathcal{KL}$-stability)

System $\dot{x} = f(x)$ is said to be $\mathcal{KL}$*-stable* if there exists $\delta > 0$ and $\beta \in \mathcal{KL}$ such that if $|x(0)| \leq \delta$ then for all $t \geq 0$,

$$|x(t)| \leq \beta(|x(0)|, t). \tag{5}$$

Proposition

The origin is asymptotically stable if and only if it is $\mathcal{KL}$-stable.

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Proposition

The origin is asymptotically stable if and only if it is $\mathcal{KL}$-stable.

Definition (Exponential stability)

The origin is *exponentially stable* for $\dot{x} = f(x)$ if there exist $\delta, \lambda, M > 0$ such that if $|x(0)| \leq \delta$ then for all $t \geq 0$,

$$|x(t)| \leq M|x(0)|e^{-\lambda t}. \tag{6}$$

Note that:

- Exponential stability $\overset{\Rightarrow}{\not\Leftarrow}$ Asymptotic stability
- Exponential stability corresponds to $\mathcal{KL}$-stability where $\beta \in \mathcal{KL}$ is of the form

$$\beta(s,t) = Mse^{-\lambda t}, \qquad s,t \geq 0.$$

Exercise:

- Show that the origin for $\dot{x} = -x$ is exponentially stable.
- Show that the origin for $\dot{x} = -x^3$ is asymptotically stable but not exponentially stable.

Stability Notions (Local versus global results)

Definition (Stability)

The origin is *Lyapunov stable* (or simply *stable*) for system $\dot{x} = f(x)$ if, for any $\varepsilon > 0$ there exists $\delta > 0$ (possibly dependent on ε) such that if $|x(0)| \leq \delta$ then, for all $t \geq 0$, $|x(t)| \leq \varepsilon$.

Definition (Global attractivity)

The origin is *globally attractive* for $\dot{x} = f(x)$ if $\forall\, x(0) \in \mathbb{R}^n$,

$$\lim_{t\to\infty} x(t) = 0.$$

Definition (Local attractivity)

The origin is *locally attractive* for $\dot{x} = f(x)$ if there exists $\gamma > 0$, so that $\forall\, x(0) \in \mathcal{B}_\gamma(0)$,

$$\lim_{t\to\infty} x(t) = 0.$$

Definition (Global $\mathcal{KL}$-stability)

System $\dot{x} = f(x)$ is *globally $\mathcal{KL}$-stable* if $|x(t)| \leq \beta(|x(0)|, t)$ holds $\forall\, x(0) \in \mathbb{R}^n$ and $\forall\, t \geq 0$.

Definition (Local $\mathcal{KL}$-stability)

System $\dot{x} = f(x)$ is *locally $\mathcal{KL}$-stable* if $|x(t)| \leq \beta(|x(0)|, t)$ holds $\forall\, x(0) \in \mathcal{B}_\gamma(0)$, $\gamma > 0$ and $\forall t \geq 0$

Definition (Global exponential stability)

The origin is *globally exponentially stable* for $\dot{x} = f(x)$ if there exist $M, \lambda > 0$ such that

$$|x(t)| \leq M|x(0)|e^{\lambda t} \qquad \forall x(0) \in \mathbb{R}^n, \quad \forall t \geq 0$$

Definition (Local exponential stability)

The origin is *locally exponentially stable* for $\dot{x} = f(x)$ if there exist $M, \lambda > 0$ and $\gamma > 0$ such that

$$|x(t)| \leq M|x(0)|e^{\lambda t} \qquad \forall x(0) \in \mathcal{B}_\gamma(0), \quad \forall t \geq 0$$

Stability Notions (Time-Varying Systems*)

So far: $\dot{x} = f(x)$, $x_0 \in \mathbb{R}^n$, $t \geq t_0 \geq 0$.
Exponential stability (depends on elapsed time):

$$|x(t)| \leq M|x(t_0)|e^{-\lambda(t-t_0)}, \quad t \geq t_0.$$

(without loss of generality $t_0 = 0$.)

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Now consider:

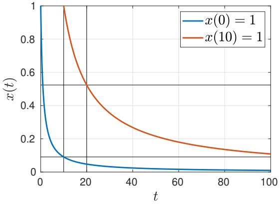
$$\dot{x} = f(t, x), \qquad x(t_0) \in \mathbb{R}^n,\ t \geq t_0 \geq 0. \tag{7}$$

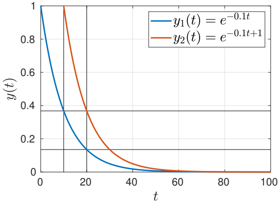
Example:

$$\dot{x} = -\frac{x}{t+1}, \quad x(t_0) \in \mathbb{R},\ t \geq t_0 \geq 0.$$

with solution

$$x(t) = x(t_0)\frac{t_0 + 1}{t + 1}.$$





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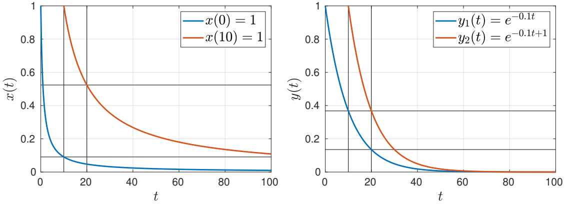

Definition (Stability)

The origin is *stable* for system (7) if, for any $\varepsilon > 0$ there exists $\delta(t_0) > 0$ such that if $|x(t_0)| \leq \delta(t_0)$ then, for all $t \geq t_0$, $|x(t)| \leq \varepsilon$.
If $\delta(t_0)$ can be chosen independent of t_0, then the origin is *uniformly stable* for system (7).

For the example: Suppose we are given $\varepsilon > 0$. Then if

$$|x(t_0)| \leq \frac{\varepsilon}{t_0+1} \doteq \delta(t_0)$$

then $|x(t)| = |x(t_0)|\frac{t_0+1}{t+1} \leq \frac{\varepsilon}{t_0+1}\frac{t_0+1}{t+1} \leq \varepsilon \ \forall\, t \geq t_0$.

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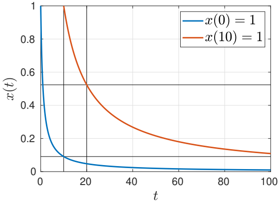

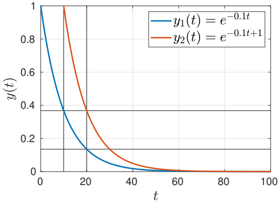

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$$|x(t_0)| \leq \frac{\varepsilon}{t_0+1} \doteq \delta(t_0)$$

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Definition ($\mathcal{KL}$-stability)

System (7) is said to be *(globally) $\mathcal{KL}$-stable* if for each $t_0 \geq 0$ there exists $\beta_{t_0} \in \mathcal{KL}$ such that for all $x(t_0) \in \mathbb{R}^n$ and $t \geq t_0$, $|x(t)| \leq \beta_{t_0}(|x(t_0)|, t - t_0)$.
If $\beta_{t_0} \in \mathcal{KL}$ can be chosen independent of t_0, then (7) is said to be uniformly globally $\mathcal{KL}$-stable.

In the example: $\beta_{t_0}(s, \tau) = s\frac{t_0+1}{\tau+t_0+1}$

Section 2

Comparison Principle

Comparison Principle

Lemma

For any $\rho \in \mathcal{P}$ there exists $\beta \in \mathcal{KL}$ such that if $y(\cdot)$ is any locally absolutely continuous function defined on some interval $[0, T]$ with $y(t) \geq 0$ for all $t \in [0, T]$, and if $y(\cdot)$ satisfies the differential inequality

$$\dot{y}(t) \leq -\rho(y(t))$$

for almost all $t \in [0, T]$ with $y(0) = y_0 \geq 0$ then

$$y(t) \leq \beta(y_0, t), \quad \forall t \in [0, T].$$

Lemma

Consider the scalar differential equation $\dot{\psi} = g(\psi)$, $\psi(0) = \psi_0 \in \mathbb{R}$. Let $[0, T)$ be the maximal interval of existence of the solution $\psi(t)$. Let $\phi(t)$ be a continuously differentiable function that satisfies

$$\dot{\phi}(t) \leq g(\phi(t)), \quad \phi(0) \leq \psi(0).$$

Then $\phi(t) \leq \psi(t)$ for all $t \in [0, T)$.

Comparison Principle

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Example

Consider:

$$\dot{x} = -(1+x^2)x, \qquad x(0) = a \in \mathbb{R} \tag{8}$$

Let

$$v(t) = x(t)^2.$$

Then:

$$\begin{aligned} \dot{v}(t) &= 2x(t)\dot{x}(t) = -2x(t)^2 - 2x(t)^4 \\ &\leq -2x(t)^2 = -2v(t). \end{aligned}$$

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$$\begin{aligned} \dot{v}(t) &= 2x(t)\dot{x}(t) = -2x(t)^2 - 2x(t)^4 \\ &\leq -2x(t)^2 = -2v(t). \end{aligned}$$

Define:

$$\dot{\psi} = -2\psi, \qquad \psi(0) = a^2, \tag{9}$$

with solution

$$\psi(t) = a^2 e^{-2t}.$$

Then:

$$|x(t)| = \sqrt{v(t)} \leq \sqrt{\psi(t)} = |a|e^{-t}.$$

Comparison Principle

Lemma

For any $\rho \in \mathcal{P}$ there exists $\beta \in \mathcal{KL}$ such that if $y(\cdot)$ is any locally absolutely continuous function defined on some interval $[0,T]$ with $y(t) \geq 0$ for all $t \in [0,T]$, and if $y(\cdot)$ satisfies the differential inequality

$$\dot{y}(t) \leq -\rho(y(t))$$

for almost all $t \in [0,T]$ with $y(0) = y_0 \geq 0$ then

$$y(t) \leq \beta(y_0, t), \quad \forall t \in [0,T].$$

Lemma

Consider the scalar differential equation $\dot{\psi} = g(\psi)$, $\psi(0) = \psi_0 \in \mathbb{R}$. Let $[0,T)$ be the maximal interval of existence of the solution $\psi(t)$. Let $\phi(t)$ be a continuously differentiable function that satisfies

$$\dot{\phi}(t) \leq g(\phi(t)), \quad \phi(0) \leq \psi(0).$$

Then $\phi(t) \leq \psi(t)$ for all $t \in [0,T)$.

Example

Consider:

$$\dot{x} = -(1+x^2)x, \qquad x(0) = a \in \mathbb{R} \tag{8}$$

Let

$$v(t) = x(t)^2.$$

Then:

$$\begin{aligned}\dot{v}(t) &= 2x(t)\dot{x}(t) = -2x(t)^2 - 2x(t)^4 \\ &\leq -2x(t)^2 = -2v(t).\end{aligned}$$

Define:

$$\dot{\psi} = -2\psi, \qquad \psi(0) = a^2, \tag{9}$$

with solution

$$\psi(t) = a^2 e^{-2t}.$$

Then:

$$|x(t)| = \sqrt{v(t)} \leq \sqrt{\psi(t)} = |a|e^{-t}.$$

$\rightsquigarrow$ origin of (9) asymp. stable $\Rightarrow$ origin of (8) is asymp. stable

Section 3

Stability by Lyapunov's Second Method

Stability by Lypunov's Second Method

Theorem (Lyapunov stability theorem)

Given $\dot{x} = f(x)$ *with* $f(0) = 0$*, and a domain* $\mathcal{D} \subset \mathbb{R}^n$*, suppose there exists a continuously differentiable function* $V : \mathcal{D} \to \mathbb{R}_{\geq 0}$ *and* $\alpha_1, \alpha_2 \in \mathcal{K}$ *such that, for all* $x \in \mathcal{D}$*,*

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \qquad \text{and} \qquad \langle \nabla V(x), f(x) \rangle \leq 0.$$

Then the origin is stable. If, additionally, $\mathcal{D} = \mathbb{R}^n$ *and* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$*, then the origin is globally stable.*

Stability by Lypunov's Second Method

Theorem (Lyapunov stability theorem)

Given $\dot{x} = f(x)$ with $f(0) = 0$, and a domain $\mathcal{D} \subset \mathbb{R}^n$, suppose there exists a continuously differentiable function $V : \mathcal{D} \to \mathbb{R}_{\geq 0}$ and $\alpha_1, \alpha_2 \in \mathcal{K}$ such that, for all $x \in \mathcal{D}$,

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Then the origin is stable. If, additionally, $\mathcal{D} = \mathbb{R}^n$ and $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, then the origin is globally stable.

Theorem (Asymptotic stability theorem)

Given $\dot{x} = f(x)$ with $f(0) = 0$, and a domain $\mathcal{D} \subset \mathbb{R}^n$, suppose there exists a continuously differentiable function $V : \mathcal{D} \to \mathbb{R}_{\geq 0}$, $\alpha_1, \alpha_2 \in \mathcal{K}$, and $\rho \in \mathcal{P}$ such that, for all $x \in \mathcal{D}$,

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \quad \text{and} \quad \langle \nabla V(x), f(x) \rangle \leq -\rho(|x|).$$

Then the origin is asymptotically stable. If, additionally, $\mathcal{D} = \mathbb{R}^n$ and $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, then the origin is globally asymptotically stable.

Stability by Lypunov's Second Method

Theorem (Lyapunov stability theorem)

Given $\dot{x} = f(x)$ with $f(0) = 0$, and a domain $\mathcal{D} \subset \mathbb{R}^n$, suppose there exists a continuously differentiable function $V : \mathcal{D} \to \mathbb{R}_{\geq 0}$ and $\alpha_1, \alpha_2 \in \mathcal{K}$ such that, for all $x \in \mathcal{D}$,

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \quad \text{and} \quad \langle \nabla V(x), f(x) \rangle \leq 0.$$

Then the origin is stable. If, additionally, $\mathcal{D} = \mathbb{R}^n$ and $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, then the origin is globally stable.

Theorem (Asymptotic stability theorem)

Given $\dot{x} = f(x)$ with $f(0) = 0$, and a domain $\mathcal{D} \subset \mathbb{R}^n$, suppose there exists a continuously differentiable function $V : \mathcal{D} \to \mathbb{R}_{\geq 0}$, $\alpha_1, \alpha_2 \in \mathcal{K}$, and $\rho \in \mathcal{P}$ such that, for all $x \in \mathcal{D}$,

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \quad \text{and} \quad \langle \nabla V(x), f(x) \rangle \leq -\rho(|x|).$$

Then the origin is asymptotically stable. If, additionally, $\mathcal{D} = \mathbb{R}^n$ and $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, then the origin is globally asymptotically stable.

Theorem (Exponential stability theorem)

Given $\dot{x} = f(x)$ with $f(0) = 0$, and a domain $\mathcal{D} \subset \mathbb{R}^n$, suppose there exist a continuously differentiable function $V : \mathcal{D} \to \mathbb{R}_{\geq 0}$ and constants $\lambda_1, \lambda_2, c > 0$ and $p \geq 1$ such that, for all $x \in \mathcal{D}$

$$\lambda_1 |x|^p \leq V(x) \leq \lambda_2 |x|^p \quad \text{and} \quad \langle \nabla V(x), f(x) \rangle \leq -cV(x).$$

Then the origin is exponentially stable. If, additionally, $\mathcal{D} = \mathbb{R}^n$, then the origin is globally exponentially stable.

Stability by Lypunov's Second Method

Theorem (Lyapunov stability theorem)

Given $\dot{x} = f(x)$ with $f(0) = 0$, and a domain $\mathcal{D} \subset \mathbb{R}^n$, suppose there exists a continuously differentiable function $V : \mathcal{D} \to \mathbb{R}_{\geq 0}$ and $\alpha_1, \alpha_2 \in \mathcal{K}$ such that, for all $x \in \mathcal{D}$,

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \quad \text{and} \quad \langle \nabla V(x), f(x) \rangle \leq 0.$$

Then the origin is stable. If, additionally, $\mathcal{D} = \mathbb{R}^n$ and $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, then the origin is globally stable.

Theorem (Asymptotic stability theorem)

Given $\dot{x} = f(x)$ with $f(0) = 0$, and a domain $\mathcal{D} \subset \mathbb{R}^n$, suppose there exists a continuously differentiable function $V : \mathcal{D} \to \mathbb{R}_{\geq 0}$, $\alpha_1, \alpha_2 \in \mathcal{K}$, and $\rho \in \mathcal{P}$ such that, for all $x \in \mathcal{D}$,

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \quad \text{and} \quad \langle \nabla V(x), f(x) \rangle \leq -\rho(|x|).$$

Then the origin is asymptotically stable. If, additionally, $\mathcal{D} = \mathbb{R}^n$ and $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, then the origin is globally asymptotically stable.

Theorem (Exponential stability theorem)

Given $\dot{x} = f(x)$ with $f(0) = 0$, and a domain $\mathcal{D} \subset \mathbb{R}^n$, suppose there exist a continuously differentiable function $V : \mathcal{D} \to \mathbb{R}_{\geq 0}$ and constants $\lambda_1, \lambda_2, c > 0$ and $p \geq 1$ such that, for all $x \in \mathcal{D}$

$$\lambda_1 |x|^p \leq V(x) \leq \lambda_2 |x|^p \quad \text{and} \quad \langle \nabla V(x), f(x) \rangle \leq -cV(x).$$

Then the origin is exponentially stable. If, additionally, $\mathcal{D} = \mathbb{R}^n$, then the origin is globally exponentially stable.

Interpretation:

- The time derivative of the "generalized energy function" V does not increase over time

$$\frac{d}{dt} V(x(t)) = \langle \nabla V(x), f(x) \rangle$$

- Stability of the origin can be concluded without knowledge of the solution.
- The theorems represent a sufficient condition (i.e., if ... then ...)

Stability by Lyapunov's Second Method (Example)

Pendulum dynamics:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\tfrac{g}{\ell}\sin x_1 - \tfrac{k}{m}x_2,$$

Total energy: $V : \mathcal{D} \to \mathbb{R}_{\geq 0}$, ($\mathcal{D} = (-\tfrac{\pi}{2}, \tfrac{\pi}{2}) \times \mathbb{R}$)

$$V(x) = mg\ell(1-\cos x_1) + \tfrac{1}{2}m\ell^2 x_2^2$$

Time derivative of candidate Lyapunov function (for $k = 0$):

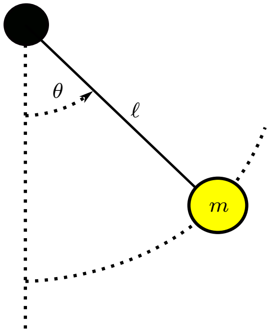
$$\langle \nabla V(x), f(x) \rangle = \begin{bmatrix} mg\ell \sin x_1 & m\ell^2 x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ -\tfrac{g}{\ell}\sin x_1 \end{bmatrix}$$
$$= mg\ell x_2 \sin x_1 - mg\ell x_2 \sin x_1 = 0 \leq 0 \quad \forall x \in \mathcal{D}$$

(Show that $\alpha_1, \alpha_2 \in \mathcal{K}$ with

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$

for all x in a neighborhood around the origin exist.)

$\rightsquigarrow$ Stability of the origin follows.



Stability by Lyapunov's Second Method (Example)

Pendulum dynamics (focus on downward equilibrium):

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\tfrac{g}{\ell}\sin x_1 - \tfrac{k}{m}x_2,\end{aligned}$$

For $k > 0$ consider the function:

$$V(x) = \tfrac{1}{2}\left(ax_1^2 + bx_1x_2 + x_2^2\right) + \tfrac{g}{\ell}(1-\cos x_1),$$

for $a, b > 0$ to be determined.
We compute the inner product

$$\langle \nabla V(x), f(x)\rangle = \begin{bmatrix} ax_1 + \frac{b}{2}x_2 + \frac{g}{\ell}\sin x_1 \\ \frac{b}{2}x_1 + x_2 \end{bmatrix}^T \begin{bmatrix} x_2 \\ -\frac{g}{\ell}\sin x_1 - \frac{k}{m}x_2 \end{bmatrix}$$

$$\begin{aligned}&= ax_1x_2 + \tfrac{b}{2}x_2^2 + \tfrac{g}{\ell}x_2\sin x_1\\ &\qquad - \tfrac{b}{2}\tfrac{g}{\ell}x_1\sin x_1 - \tfrac{b}{2}\tfrac{k}{m}x_1x_2 - \tfrac{g}{\ell}x_2\sin x_1 - \tfrac{k}{m}x_2^2\\ &= -\tfrac{b}{2}\tfrac{g}{\ell}x_1\sin x_1 - \left(\tfrac{k}{m} - \tfrac{b}{2}\right)x_2^2 + \left(a - \tfrac{b}{2}\tfrac{k}{m}\right)x_1x_2.\end{aligned}$$

Stability by Lyapunov's Second Method (Example)

Pendulum dynamics (focus on downward equilibrium):

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{\ell}\sin x_1 - \frac{k}{m}x_2,$$

For $k > 0$ consider the function:

$$V(x) = \frac{1}{2}\left(ax_1^2 + bx_1x_2 + x_2^2\right) + \frac{g}{\ell}(1 - \cos x_1),$$

for $a, b > 0$ to be determined.
We compute the inner product

$$\langle \nabla V(x), f(x)\rangle = \begin{bmatrix} ax_1 + \frac{b}{2}x_2 + \frac{g}{\ell}\sin x_1 \\ \frac{b}{2}x_1 + x_2 \end{bmatrix}^T \begin{bmatrix} x_2 \\ -\frac{g}{\ell}\sin x_1 - \frac{k}{m}x_2 \end{bmatrix}$$
$$= ax_1x_2 + \frac{b}{2}x_2^2 + \frac{g}{\ell}x_2\sin x_1$$
$$\quad - \frac{b}{2}\frac{g}{\ell}x_1\sin x_1 - \frac{b}{2}\frac{k}{m}x_1x_2 - \frac{g}{\ell}x_2\sin x_1 - \frac{k}{m}x_2^2$$
$$= -\frac{b}{2}\frac{g}{\ell}x_1\sin x_1 - \left(\frac{k}{m} - \frac{b}{2}\right)x_2^2 + \left(a - \frac{b}{2}\frac{k}{m}\right)x_1x_2.$$

Define $a = \frac{b}{2}\frac{k}{m}$ and $b = \frac{k}{m}$
(to eliminate the cross term and ensure that the coefficient of x_2^2 is negative)

Then

$$\langle \nabla V(x), f(x)\rangle = -\frac{gk}{2\ell m}x_1\sin x_1 - \frac{k}{2m}x_2^2 < 0 \quad \forall x \in \mathcal{D}\backslash\{0\}$$

Stability by Lyapunov's Second Method (Example)

Pendulum dynamics (focus on downward equilibrium):

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{\ell}\sin x_1 - \frac{k}{m}x_2,$$

For $k > 0$ consider the function:

$$V(x) = \frac{1}{2}\left(ax_1^2 + bx_1x_2 + x_2^2\right) + \frac{g}{\ell}(1-\cos x_1),$$

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$$\langle \nabla V(x), f(x)\rangle = \begin{bmatrix} ax_1 + \frac{b}{2}x_2 + \frac{g}{\ell}\sin x_1 \\ \frac{b}{2}x_1 + x_2 \end{bmatrix}^T \begin{bmatrix} x_2 \\ -\frac{g}{\ell}\sin x_1 - \frac{k}{m}x_2 \end{bmatrix}$$
$$= ax_1x_2 + \frac{b}{2}x_2^2 + \frac{g}{\ell}x_2\sin x_1$$
$$\quad - \frac{b}{2}\frac{g}{\ell}x_1\sin x_1 - \frac{b}{2}\frac{k}{m}x_1x_2 - \frac{g}{\ell}x_2\sin x_1 - \frac{k}{m}x_2^2$$
$$= -\frac{b}{2}\frac{g}{\ell}x_1\sin x_1 - \left(\frac{k}{m} - \frac{b}{2}\right)x_2^2 + \left(a - \frac{b}{2}\frac{k}{m}\right)x_1x_2.$$

Define $a = \frac{b}{2}\frac{k}{m}$ and $b = \frac{k}{m}$
(to eliminate the cross term and ensure that the coefficient of x_2^2 is negative)

Then

$$\langle \nabla V(x), f(x)\rangle = -\frac{gk}{2\ell m}x_1\sin x_1 - \frac{k}{2m}x_2^2 < 0 \quad \forall x \in \mathcal{D}\backslash\{0\}$$

Check that V is positive definite:

$$V(x) = \frac{1}{2}x^TPx + \frac{g}{\ell}(1-\cos(x_1))$$

$$P = \begin{bmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left(\frac{k}{m}\right)^2 & \frac{1}{2}\frac{k}{m} \\ \frac{1}{2}\frac{k}{m} & 1 \end{bmatrix}.$$

The matrix P is positive definite since

$$\frac{1}{2}\left(\frac{k}{m}\right)^2 > 0, \qquad \frac{1}{2}\left(\frac{k}{m}\right)^2 - \frac{1}{4}\left(\frac{k}{m}\right)^2 > 0$$

(leading principal minors are all positive)

Stability by Lyapunov's Second Method (Example)

Pendulum dynamics (focus on downward equilibrium):

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{\ell}\sin x_1 - \frac{k}{m}x_2,$$

For $k > 0$ consider the function:

$$V(x) = \frac{1}{2}\left(ax_1^2 + bx_1x_2 + x_2^2\right) + \frac{g}{\ell}(1-\cos x_1),$$

for $a, b > 0$ to be determined.
We compute the inner product

$$\langle \nabla V(x), f(x)\rangle = \begin{bmatrix} ax_1 + \frac{b}{2}x_2 + \frac{g}{\ell}\sin x_1 \\ \frac{b}{2}x_1 + x_2 \end{bmatrix}^T \begin{bmatrix} x_2 \\ -\frac{g}{\ell}\sin x_1 - \frac{k}{m}x_2 \end{bmatrix}$$

$$= ax_1x_2 + \frac{b}{2}x_2^2 + \frac{g}{\ell}x_2\sin x_1$$
$$\quad - \frac{b}{2}\frac{g}{\ell}x_1\sin x_1 - \frac{b}{2}\frac{k}{m}x_1x_2 - \frac{g}{\ell}x_2\sin x_1 - \frac{k}{m}x_2^2$$
$$= -\frac{b}{2}\frac{g}{\ell}x_1\sin x_1 - \left(\frac{k}{m} - \frac{b}{2}\right)x_2^2 + \left(a - \frac{b}{2}\frac{k}{m}\right)x_1x_2.$$

Define $a = \frac{b}{2}\frac{k}{m}$ and $b = \frac{k}{m}$
(to eliminate the cross term and ensure that the coefficient of x_2^2 is negative)

Then

$$\langle \nabla V(x), f(x)\rangle = -\frac{gk}{2\ell m}x_1\sin x_1 - \frac{k}{2m}x_2^2 < 0 \ \ \forall x \in \mathcal{D}\backslash\{0\}$$

Check that V is positive definite:

$$V(x) = \frac{1}{2}x^TPx + \frac{g}{\ell}(1-\cos(x_1))$$

$$P = \begin{bmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left(\frac{k}{m}\right)^2 & \frac{1}{2}\frac{k}{m} \\ \frac{1}{2}\frac{k}{m} & 1 \end{bmatrix}.$$

The matrix P is positive definite since

$$\frac{1}{2}\left(\frac{k}{m}\right)^2 > 0, \qquad \frac{1}{2}\left(\frac{k}{m}\right)^2 - \frac{1}{4}\left(\frac{k}{m}\right)^2 > 0$$

(leading principal minors are all positive)

$\rightsquigarrow$ V is a Lyapunov function and asymptotic stability follows

Advantages and disadvantages:

- No solution of $\dot{x} = f(x)$ necessary. ✓
- How to find Lyapunov function V?

Stability by Lyapunov's Second Method (Proof: Lyapunov function $\Rightarrow$ asymptotic stability)

For simplicity, assume that $\mathcal{D} = \mathbb{R}^n$ (i.e., we show global asymptotic stability).
It holds that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|).$$

For $\rho \in \mathcal{P}$, there exist $\hat{\alpha} \in \mathcal{K}_\infty$, $\sigma \in \mathcal{L}$ so that

$$\rho(|x|) > \hat{\alpha}(|x|)\sigma(|x|).$$

The decrease condition of the Lyapunov function implies:

$$\begin{aligned}\langle \nabla V(x), f(x) \rangle &\leq -\rho(|x|) \leq -\hat{\alpha}(|x|)\sigma(|x|) \\ &\leq -\hat{\alpha}(\alpha_2^{-1}(V(x)))\sigma(\alpha_1^{-1}(V(x))) \\ &\leq -\hat{\rho}(V(x))\end{aligned}$$

where

$$\hat{\rho}(s) \doteq \hat{\alpha}(\alpha_2^{-1}(s))\sigma(\alpha_1^{-1}(s)), \quad \forall s \in \mathbb{R}_{\geq 0}, \qquad \hat{\rho} \in \mathcal{P}.$$

Hence

$$\tfrac{d}{dt} V(x(t)) = \langle \nabla V(x(t)), f(x(t)) \rangle \leq -\hat{\rho}(V(x(t)))$$

Then there exists $\hat{\beta} \in \mathcal{KL}$ (see Comparison Principle) so that

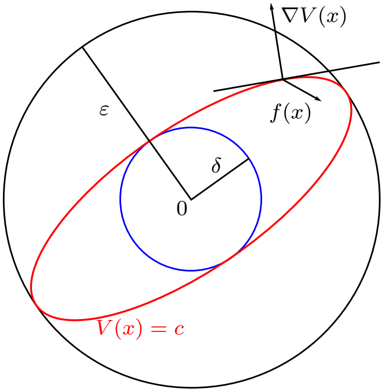
$$V(x(t)) \leq \hat{\beta}(V(x(0)), t), \quad \forall t \geq 0.$$

Then

$$\alpha_1(|x(t)|) \leq V(x(t)) \leq \hat{\beta}(V(x(0)), t) \leq \hat{\beta}(\alpha_2(|x(0)|), t)$$

and with the $\mathcal{KL}$ function $\beta(s,t) \doteq \alpha_1^{-1}(\hat{\beta}(\alpha_2(s), t))$ for all $s, t \in \mathbb{R}_{\geq 0}$, global $\mathcal{KL}$-stability of the system follows.

Stability by Lyapunov's Second Method (Proof: Lyapunov function $\Rightarrow$ asymptotic stability)



Given $\varepsilon > 0$, define

$$\overline{\mathcal{B}}_\varepsilon = \{x \in \mathbb{R}^n : |x| \leq \varepsilon\}.$$

Without loss of generality, $\overline{\mathcal{B}}_\varepsilon \subset \mathcal{D}$
(otherwise shrink ε)

Let $a = \min_{|x|=\varepsilon} V(x)$ and take $c \in (0, a)$. Define

$$\Omega_c = \{x \in \mathcal{D} : V(x) \leq c\}$$

and observe that $\Omega_c \subset \mathcal{B}_\varepsilon$.
For x on the boundary of Ω_c, $\nabla V(x)$ is the outward-facing normal vector.
The decrease condition implies

$$\langle \nabla V(x), f(x) \rangle = |\nabla V(x)||f(x)| \cos(\theta) \leq 0,$$

i.e., $|\theta| \geq \frac{\pi}{2}$
Thus Ω_c is *invariant*; i.e., solutions starting in Ω_c will always remain in Ω_c.
ε-δ game: We choose $\delta > 0$ so that

$$\overline{\mathcal{B}}_\delta = \{x \in \mathcal{D} : |x| \leq \delta\} \subset \Omega_c$$

Therefore, if $|x(0)| \leq \delta$ then $x(0) \in \mathcal{B}_\delta \subset \Omega_c$ and, forward invariance implies $x(t) \in \Omega_c \subset \overline{\mathcal{B}}_\varepsilon$
Thus $|x(t)| \leq \varepsilon$ for all $t \geq 0$ (i.e., stability).
To prove asymptotic stability we use

$$\langle \nabla V(x), f(x) \rangle = |\nabla V(x)||f(x)| \cos(\theta) < 0 \qquad \text{if } x \neq 0$$

Stability by Lyapunov's Second Method (Additional results)

Theorem (Rescaling of Lyapunov functions)

Let $\alpha \in \mathcal{K}_\infty$ be continuously differentiable on $\mathbb{R}_{>0}$ and $\alpha'(s) > 0$ for all $s > 0$. If $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a Lyapunov function for $\dot{x} = f(x)$, then $W : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ defined by

$$W(x) \doteq \alpha(V(x)), \quad \forall x \in \mathbb{R}^n$$

is also a Lyapunov function for $\dot{x} = f(x)$.

Theorem (Exp. decreasing Lyapunov functions)

If there exists a Lyapunov function for system $\dot{x} = f(x)$ satisfying

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \quad \text{and} \quad \langle \nabla V(x), f(x) \rangle \leq -\rho(|x|).$$

then there exist a continuously differentiable function $W : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ with $W(0) = 0$ and $\hat{\alpha}_1, \hat{\alpha}_2 \in \mathcal{K}_\infty$ so that, for all $x \in \mathbb{R}^n$,

$$\hat{\alpha}_1(|x|) \leq W(x) \leq \hat{\alpha}_2(|x|) \quad \text{and} \quad \langle \nabla W(x), f(x) \rangle \leq -W(x)$$

Stability by Lyapunov's Second Method (Additional results)

Theorem (Rescaling of Lyapunov functions)

Let $\alpha \in \mathcal{K}_\infty$ be continuously differentiable on $\mathbb{R}_{>0}$ and $\alpha'(s) > 0$ for all $s > 0$. If $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a Lyapunov function for $\dot{x} = f(x)$, then $W : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ defined by

$$W(x) \doteq \alpha(V(x)), \quad \forall x \in \mathbb{R}^n$$

is also a Lyapunov function for $\dot{x} = f(x)$.

Theorem (Exp. decreasing Lyapunov functions)

If there exists a Lyapunov function for system $\dot{x} = f(x)$ satisfying

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \quad \textit{and} \quad \langle \nabla V(x), f(x) \rangle \leq -\rho(|x|).$$

then there exist a continuously differentiable function $W : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ with $W(0) = 0$ and $\hat{\alpha}_1, \hat{\alpha}_2 \in \mathcal{K}_\infty$ so that, for all $x \in \mathbb{R}^n$,

$$\hat{\alpha}_1(|x|) \leq W(x) \leq \hat{\alpha}_2(|x|) \quad \textit{and} \quad \langle \nabla W(x), f(x) \rangle \leq -W(x)$$

These results imply that

- If we know one Lyapunov function we can construct infinitely many
- If we know one Lyapunov function we can construct a Lyapunov function which decreases exponentially. (This follows from $\dot{w} \leq -w \Rightarrow w(t) \leq w(0)e^{-t}$, comparison principle)
- This does not imply that $|x(t)|$ decreases exponentially (i.e., it does not imply exponential stability)!

Stability by Lyapunov's Second Method (Time-Varying Systems)

Theorem (Lyapunov uniform asymptotic stab.)

Given the time-varying system $\dot{x} = f(t,x)$ *with* $f(t,0) = 0$ *for all* $t \geq t_0 \geq 0$. *If there exist a continuously differentiable function* $V : \mathbb{R}_{\geq 0} \times \mathcal{D} \to \mathbb{R}_{\geq 0}$, *and functions* $\alpha_1, \alpha_2 \in \mathcal{K}$ *and* $\rho \in \mathcal{P}$ *such that, for all* $x \in \mathcal{D}$ *and* $t \geq t_0 \geq 0$,

$$\alpha_1(|x|) \leq V(t,x) \leq \alpha_2(|x|) \qquad \textit{and}$$

$$\frac{d}{dt}V(t,x) = \nabla_t V(t,x) + \langle \nabla_x V(t,x), f(t,x) \rangle \leq -\rho(|x|)$$

then the origin is uniformly asymptotically stable.
If additionally $\mathcal{D} = \mathbb{R}^n$ *and* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, *then the origin is uniformly globally asymptotically stable.*

Compared to the time-invariant setting

- time varying-Lyapunov functions need to be considered
- the bound $V(t,x) \leq \alpha_2(|x|)$ is restrictive
  (This property is called decrescent)

Stability by Lyapunov's Second Method (Instability)

Theorem (Lyapunov theorem for instability)

Given $\dot{x} = f(x)$ with $f(0) = 0$, suppose there exist a continuously differentiable positive definite function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and an $\varepsilon > 0$ such that

$$\langle \nabla V(x), f(x) \rangle > 0$$

for all $x \in \mathcal{B}_\varepsilon \backslash \{0\}$. Then the origin is unstable.
(In fact, the origin is completely unstable.)

Cannot be used to show that the origin of $\dot{x}_1 = x_1 \; \dot{x}_2 = -x_2$ is unstable. (Not uncommon that a system exhibits stable behavior in some directions and unstable in others.)

Theorem (Chetaev's theorem)

Given $\dot{x} = f(x)$ with $f(0) = 0$, let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function with $V(0) = 0$ and $\mathcal{O}_r = \{x \in \mathcal{B}_r(0) | \; V(x) > 0\} \neq \emptyset$ for all $r > 0$. If for certain $r > 0$,

$$\langle \nabla V(x), f(x) \rangle > 0, \qquad \forall \, x \in \mathcal{O}_r$$

then the origin is unstable.

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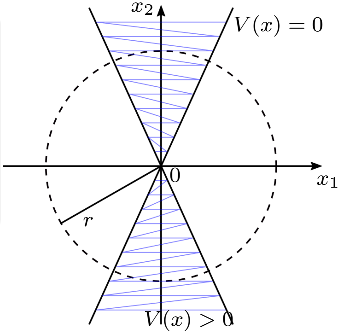

$$\dot{x}_1 = x_1$$
$$\dot{x}_2 = -x_2$$
$$V(x) = \tfrac{1}{2}x_1^2 - \tfrac{1}{2}x_2^2$$

It holds that:

- $V(x) > 0$ for all $|x_1| > |x_2|$
- $\mathcal{O}_r = \{x \in \mathcal{B}_r(0) | \, V(x) > 0\} \neq \emptyset$ for all $r > 0$
- For all $x \in \mathcal{O}_r$ (in fact, for all $x \in \mathbb{R}^2 \backslash \{0\}$):

$$\langle \nabla V(x), f(x) \rangle = [x_1 \quad -x_2] \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix} = x_1^2 + x_2^2 > 0$$

Section 4

Region of Attraction

The Region of Attraction (Estimates using Lyapunov Functions)

Definition (Region of attraction)

Consider $\dot{x} = f(x)$ with an asymptotically stable eq. $f(x^e) = 0$, $x^e \in \mathbb{R}^n$. The region of attraction of x^e:

$$\mathcal{R}_f(x^e) = \{x \in \mathbb{R}^n |\, x(t) \to x^e \text{ as } t \to \infty, x(0) = x\}\,.$$

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Properties:

- The region of attraction is an open, connected, invariant set
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Example: Consider the system

$$\dot{x}_1 = -x_2, \qquad \dot{x}_2 = x_1 + (x_1^2 - 1)x_2$$

with locally asymptotically stable equilibrium $x^e = 0$.

Example (Lyapunov function based estimate)

The function

$$V(x) = x^T P x = x^T \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} x \qquad \text{satisfies}$$

$$0.69|x|^2 \leq \lambda_{\min}(P)|x|^2 \leq V(x) \leq \lambda_{\max}(P)|x|^2 \leq 1.81|x|^2$$

Moreover,

$$\tfrac{d}{dt} V(x) = -x_1^2 - x_2^2 - x_1^3 x_2 + 2x_2^2 x_1^2$$

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Lemma (Young's inequality)

Let $p, q \in \mathbb{R}_{>0}$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then for any $x, y \in \mathbb{R}^n$ the inequality $x^T y \leq \frac{1}{p}|x|^p + \frac{1}{q}|y|^q$ is satisfied.

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Moreover,

$$\begin{aligned} \tfrac{d}{dt}V(x) &= -x_1^2 - x_2^2 - x_1^3 x_2 + 2x_2^2 x_1^2 \\ &\leq -x_1^2 - x_2^2 + x_1^6 + \tfrac{1}{4}x_2^2 + x_1^4 + x_2^4 \\ &= -x_1^2\left(1 - x_1^2 - x_1^4\right) - x_2^2\left(\tfrac{3}{4} - x_2^2\right) \end{aligned}$$

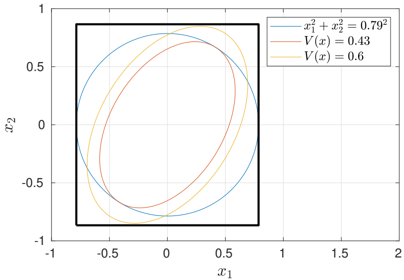
which implies that $\dot{V}(x) < 0$ whenever

$$1 - x_1^2 - x_1^4 > 0 \qquad \text{and} \qquad \tfrac{3}{4} - x_2^2 > 0.$$

The constraints can be translated into the constraints

$$\mathcal{C} = \{x \in \mathbb{R}^2 |\, -0.79 < x_1 < 0.79,\ -0.89 < x_2 < 0.89\}$$

The Region of Attraction (Estimates using Lyapunov Functions), continued



Properties:

- The set $\mathcal{C}$ is not necessarily forward invariant
- Thus, we need to define a forward invariant sublevel set.
- It holds that

$$\{x \in \mathbb{R}^2 : x^T P x \leq \lambda_{\min}\} \subset \{x \in \mathbb{R}^2 : x^T x \leq 1\},$$

$$\{x \in \mathbb{R}^2 : x^T x \leq 0.79^2\} \subset \mathcal{C}$$

  and thus $\{x \in \mathbb{R}^2 | \ x^T P x \leq 0.79^2 \lambda_{\min}\} \subset \mathcal{C}$
- We conclude $\{x \in \mathbb{R}^2 | \ x^T P x \leq 0.79^2 \lambda_{\min}\} \subset \mathcal{R}_f(0)$

Example (Lyapunov function based estimate)

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The region of attraction (Estimates in $\mathbb{R}^2$ using time reversal dynamics)

Definition (Region of attraction)

Consider $\dot{x} = f(x)$ with an asymptotically stable eq. $f(x^e) = 0$, $x^e \in \mathbb{R}^n$. The region of attraction of x^e:

$$\mathcal{R}_f(x^e) = \{x \in \mathbb{R}^n | \, x(t) \to x^e \text{ as } t \to \infty, x(0) = x\}.$$

Properties:

- The region of attraction is an open, connected, invariant set
- The calculation is far from trivial

Example: Consider the system

$$\dot{x}_1 = -x_2, \qquad \dot{x}_2 = x_1 + (x_1^2 - 1)x_2$$

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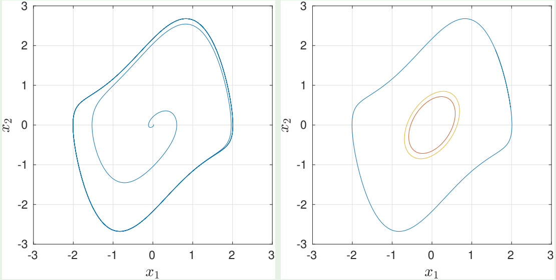
Example

Rather than considering $t \to \infty$, consider simulating backwards in time; i.e., take $t \to -\infty$. To see the effect of this, let $\tau = -t$ which implies $d\tau = -dt$ and

$$\frac{d}{d\tau}x(\tau) = -\frac{d}{dt}x(-t) = -f(x(-t)) = -f(x(\tau)).$$

In other words, simulating the system backwards in time merely requires changing the sign of the vector field.



Section 5

Converse Theorems

Converse Lyapunov Theorems

Theorem (Converse theorem; asymp. stability)

If the origin is uniformly globally asymptotically stable for $\dot{x} = f(t,x)$ then there exist a (smooth) function $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, and a function $\rho \in \mathcal{P}$ such that, for all $x \in \mathbb{R}^n$ and all $t \geq t_0 \geq 0$,

$$\alpha_1(|x|) \leq V(t,x) \leq \alpha_2(|x|) \qquad \text{and}$$
$$\nabla_t V(t,x) + \langle \nabla_x V(t,x), f(t,x) \rangle \leq -\rho(|x|).$$

- *If $f(t,x)$ is periodic in t, then there exists $V(t,x)$ periodic in t.*
- *If $f(t,x) = f(x)$ is time-invariant, then there exists $V(t,x) = V(x)$ independent of t.*

Converse Lyapunov Theorems

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$$\alpha_1(|x|) \leq V(t,x) \leq \alpha_2(|x|) \qquad \text{and}$$
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- *If $f(t,x)$ is periodic in t, then there exists $V(t,x)$ periodic in t.*
- *If $f(t,x) = f(x)$ is time-invariant, then there exists $V(t,x) = V(x)$ independent of t.*

$\rightsquigarrow$ Based on this result, is it easy to find Lyapunov functions?

Unfortunately not! Converse results for exponential stability rely on

$$V(x) = \int_0^\infty |x(\tau)| e^\tau \, d\tau, \qquad x = x(0) \in \mathbb{R}^n$$

Section 6

Invariance Theorems

Invariance Theorems (Krasovskii-LaSalle Invariance Theorem; Example 1)

Theorem (Krasovskii-LaSalle Invariance Thm.)

Suppose there exists a positive definite and continuously differentiable function $V:\mathbb{R}^n\to\mathbb{R}_{\geq 0}$ such that, for all $x\in\mathbb{R}^n$,

$$\langle\nabla V(x),f(x)\rangle\leq 0.$$

Let $S=\{x\in\mathbb{R}^n|\langle\nabla V(x),f(x)\rangle=0\}$ and suppose no solution other than the origin can stay identically in S. Then the origin is globally asymptotically stable.

Recall:

- Pendulum dynamics

$$\dot{x}_1=x_2$$
$$\dot{x}_2=-\tfrac{g}{\ell}\sin x_1-\tfrac{k}{m}x_2$$

- Total energy

$$V(x)=mg\ell(1-\cos x_1)+\tfrac{1}{2}m\ell^2x_2^2$$

Application of the Theorem:

- Time derivative of total energy:

$$\langle\nabla V(x),f(x)\rangle=-k\ell^2x_2^2.$$

- It holds that

$$\langle\nabla V(x),f(x)\rangle=0\quad\text{whenever}\quad\begin{bmatrix}x_1\\x_2\end{bmatrix}\in\mathbb{R}\times\{0\}.$$

⇝ (Thus, asymptotic stability cannot be concluded from the Lyapunov Theorem)

- (Let $\mathcal{D}$ define a neighborhood around the origin) Define

$$S=\{x\in\mathcal{D}:x_2=0\}.$$

- Note that
  - for $x_2=0$ to remain at zero, $\dot{x}_2=0$ needs to be satisfied.
  - with the dynamics, this implies $x_1=0$ and $\dot{x}_1=0$
- Hence, the only solution that can remain in S is $x_1(t)=0$, $x_2(t)=0$.

Invariance Theorems (Krasovskii-LaSalle Invariance Theorem; Example 2)

Theorem (Krasovskii-LaSalle Invariance Thm.)

Suppose there exists a positive definite and continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that, for all $x \in \mathbb{R}^n$,

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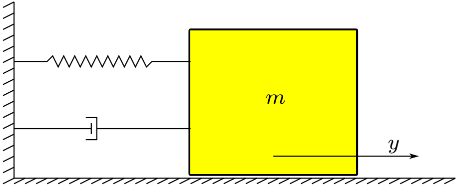

Recall: $m\ddot{y} + b\dot{y}|\dot{y}| + k_0 y + k_1 y^3 = 0.$

State space model ($x_1 = y$ and $x_2 = \dot{y}$):

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{1}{m}\left(-k_0 x_1 - k_1 x_1^3 - b x_2 |x_2|\right).$$

Consider the candidate Lyapunov function

$$V(x) = \frac{k_0}{2m} x_1^2 + \frac{k_1}{4m} x_1^4 + \frac{1}{2} x_2^2.$$

Then

$$\langle \nabla V(x), f(x) \rangle$$
$$= \frac{k_0}{m} x_1 x_2 + \frac{k_1}{m} x_1^3 x_2 - \frac{k_0}{m} x_1 x_2 - \frac{k_1}{m} x_1^3 x_2 - \frac{b}{m} x_2^2 |x_2|$$
$$= -\frac{b}{m} x_2^2 |x_2| \leq 0.$$

and $\langle \nabla V(x), f(x) \rangle = 0$ for all $x_1 \in \mathbb{R}$, $x_2 = 0$

Define $\quad S = \{x \in \mathbb{R}^2 | \; x_2 = 0\}.$

In S, $x_2 = 0$ and $\dot{x}_2 = 0$ to stay in S.

Thus $\dot{x}_1 = 0$ and

$$0 = -\frac{1}{m}(k_0 x_1 + k_1 x_1^3) \quad \Rightarrow \quad x_1 = 0 \;\text{ or }\; x_1 = \pm j\sqrt{\frac{k_0}{k_1}}.$$

Therefore, $x = 0$ is asymptotically stable.

Introduction to Nonlinear Control

Stability, control design, and estimation

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Australian National University, Canberra, Australia

Part I:

Chapter 2: Nonlinear Systems - Stability Notions

Australian National University