Introduction to Nonlinear Control

Stability, control design, and estimation

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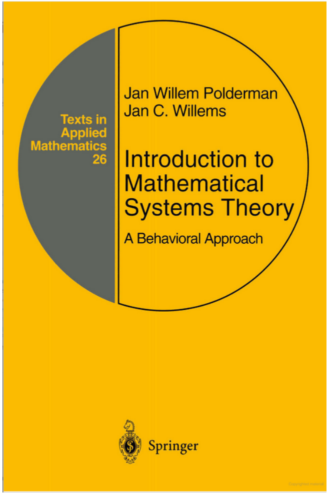
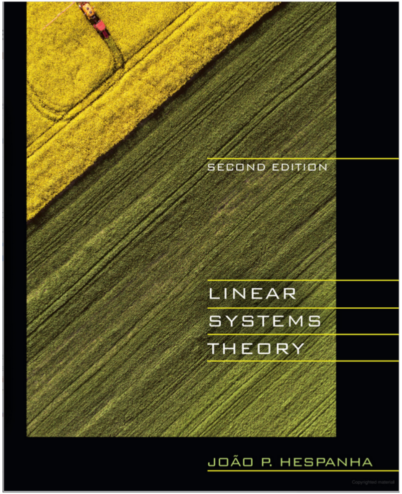
Part I:

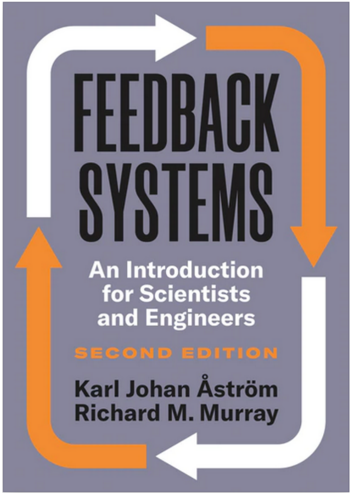
Chapter 3: Linear Systems and Linearization

Australian National University

Linear Systems and Linearization







Linear Systems and Linearization

Linear Systems Review

Simplest example ($a \in \mathbb{R}$):

$$\dot{x} = ax, \qquad x(0) = x_0 \in \mathbb{R}$$

In this case, solution is given by

$$x(t) = e^{at}x(0), \qquad t \geq 0$$

(since $\frac{d}{dt}x(t) = ax(0)e^{at} = ax(t)$)

Exponential function:

$$e^a = \sum_{k=0}^{\infty} \frac{1}{k!} a^k.$$

The origin is:

- (uniformly) globally exponentially stable if and only if $a < 0$;
- globally stable if and only if $a = 0$; and
- unstable if and only if $a > 0$.

Consider $V(x) = x^2$. If $a < 0$, it holds that

$$\langle \nabla V(x), \dot{x} \rangle = \langle 2x, ax \rangle = 2ax^2 \leq 2aV(x) \quad \forall x \in \mathbb{R}$$

$\rightsquigarrow$ V is a Lyapunov function from which global exponential stability can be concluded

Linear systems (defined through $A \in \mathbb{R}^{n \times n}$):

$$\dot{x} = Ax, \qquad x(0) = x_0 \in \mathbb{R}^n$$

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Matrix exponential:

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Linear Systems Review (Stability Properties for Linear Systems)

The matrix exponential:

- Consider $A \in \mathbb{R}^{n\times n}$ diagonalizable
- Then there exists $T \in \mathbb{C}^{n\times n}$ so that $\Lambda = T^{-1}AT \in \mathbb{C}^{n\times n}$ diagonal
- (Λ contains the eigenvalues of A)
- Observe that

$$A^k = (T\Lambda T^{-1})(T\Lambda T^{-1})\cdots(T\Lambda T^{-1}) = T\Lambda^k T^{-1}$$

- Therefore,

$$e^{At} = \sum_{k=0}^{\infty} \frac{t^k}{k!}A^k = T\left(\sum_{k=0}^{\infty}\frac{t^k}{k!}\Lambda^k\right)T^{-1}$$

$$= T\begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t}\end{bmatrix}T^{-1}$$

- It holds that $|x(t)| = |Te^{\Lambda t}T^{-1}x(0)| \overset{t\to\infty}{\to} 0$ $\forall x(0) \in \mathbb{R}^n$ if $\mathrm{Re}(\lambda_i) < 0 \ \forall\, i = 1,\dots,n$.

The matrix exponential (A not diagonalizable):

- Consider Jordan normal form (example, 2×2-block)

$$J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \quad \rightsquigarrow \quad J^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1} \\ 0 & \lambda^k \end{bmatrix}$$

- Therefore, the diagonal elements satisfy $e^{\lambda t}$ and the $(1,2)$-entry satisfies

$$\sum_{k=0}^{\infty}\frac{kt^k}{k!}\lambda^{k-1} = t\sum_{k=1}^{\infty}\frac{t^{k-1}}{(k-1)!}\lambda^{k-1} = t\sum_{\ell=0}^{\infty}\frac{t^\ell}{\ell!}\lambda^\ell = te^{\lambda t}.$$

- Finally, we can conclude

$$e^{Jt} = e^{\lambda t}\begin{bmatrix} 1 & t \\ 0 & 1\end{bmatrix}.$$

- A 3×3-block:

$$J = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \quad \rightsquigarrow \quad e^{Jt} = e^{\lambda t}\begin{bmatrix} 1 & t & \frac{t^2}{2!} \\ 0 & 1 & t \\ 0 & 0 & 1\end{bmatrix}$$

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Linear Systems Review (Stability Properties for Linear Systems, 2)

Theorem (Stability of linear systems)

For the linear system $\dot{x} = Ax$, the origin is

1. *stable if and only if the eigenvalues of A have negative or zero real parts and all the Jordan blocks corresponding to eigenvalues with zero real parts are 1×1;*
2. *unstable if and only if at least one eigenvalue of A has a positive real part or zero real part with the corresponding Jordan block larger than 1×1;*
3. *exponentially stable if and only if all the eigenvalues of A have strictly negative real parts.*

Note that for linear systems:

- It is common to say 'the linear system is asymptotically stable' (linear systems can only have 1 isolated equilibrium, i.e., the origin)
- If all eigenvalues of A have strictly negative real parts, A is said to be *Hurwitz*
- Local stability results imply global stability results
- asymptotic stability implies exponential stability

A diagonalizable:

- Therefore,

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Linear Systems Review (Quadratic Lyapunov Functions)

Notation:

- Symmetric matrices

$$\mathcal{S}^n = \{P \in \mathbb{R}^{n\times n} |\ P = P^T\}$$

- Positive (semi)definite matrices:

$$\mathcal{S}^n_{>0} = \{P \in \mathcal{S}^n |\ x^T Px > 0 \ \forall\, x \neq 0\}$$
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- Quadratic candidate Lyapunov functions:

$$V(x) = x^T Px$$

- If $P \in \mathcal{S}^n_{>0}$ then

$$0 < \lambda_{\min} x^T x \leq x^T Px \leq \lambda_{\max} x^T x, \quad \forall\, x \neq 0 \quad (1)$$

(symmetric matrices have real eigenvalues)

- Recall the condition:

$$\alpha_1(|x|^2) \leq V(x) \leq \alpha_2(|x|^2), \quad \alpha_1, \alpha_2 \in \mathcal{K}$$

Lemma

The following are equivalent:

1. $P \in \mathcal{S}^n_{>0}$;
2. *All the eigenvalues of P are positive;*
3. *The determinants of all the upper left submatrices (the so-called leading principal minors) of P are positive;*
4. *There exists a nonsingular matrix $H \in \mathbb{R}^{n\times n}$ such that $P = H^T H$.*

Theorem

For the linear system $\dot{x} = Ax$, the following are equivalent:

1. *The origin is exponentially stable;*
2. *All eigenvalues of A have strictly negative real parts;*
3. *For every $Q \in \mathcal{S}^n_{>0}$ there exists a unique $P \in \mathcal{S}^n_{>0}$, satisfying the Lyapunov equation*

$$A^T P + PA = -Q.$$

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Linear Systems Review (Quadratic Lyapunov Functions)

Proof: $Q \in \mathcal{S}^n_{>0}$, $P \in \mathcal{S}^n_{>0}$ such that $A^T P + PA = -Q \implies$ exponential stability:

- For simplicity take $Q = I$.
- Then

$$\lambda_{\min} x^T x \leq x^T P x \leq \lambda_{\max} x^T x \quad \Rightarrow \quad -x^T x \leq -\frac{1}{\lambda_{\max}} x^T P x$$

- Application of the chain rule,

$$\frac{d}{dt} V(x) = \dot{x}^T P x + x^T P \dot{x} = x^T A^T P x + x^T P A x = x^T (A^T P + PA) x = -x^T x \leq -\frac{1}{\lambda_{\max}} x^T P x = -\frac{1}{\lambda_{\max}} V(x)$$

- Comparison principle:

$$V(x(t)) \leq V(x(0)) \exp\left(-\frac{1}{\lambda_{\max}} t\right)$$

- Then

$$\lambda_{\min} |x(t)|^2 \leq V(x(t)) \leq V(x(0)) \exp\left(-\frac{1}{\lambda_{\max}} t\right) \leq \lambda_{\max} |x(0)|^2 \exp\left(-\frac{1}{\lambda_{\max}} t\right)$$

$$\Rightarrow |x(t)| \leq \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} |x(0)| \exp\left(-\frac{1}{2\lambda_{\max}} t\right)$$

$$\Rightarrow |x(t)| \leq M |x(0)| \exp(-\lambda t), \quad M, \lambda > 0 \rightsquigarrow \text{exponential stability}$$

Linear Systems Review (Quadratic Lyapunov Functions)

Proof: Exponential stability $\Longrightarrow$ For every $Q \in \mathcal{S}^n_{>0}$ there exists a unique $P \in \mathcal{S}^n_{>0}$, satisfying $A^T P + PA = -Q$:

- Given $Q \in \mathcal{S}^n_{>0}$, let
$$P = \int_0^\infty e^{A^T\tau} Q e^{A\tau} d\tau.$$
- (Note that $\|e^{A^T t} Q e^{At}\| \overset{t\to}{\longrightarrow} 0$ exponentially fast, i.e., the integral is well defined)
- It holds that
$$\tfrac{d}{dt}\left(e^{A^T t} Q e^{At}\right) = A^T e^{A^T t} Q e^{At} + e^{A^T t} Q e^{At} A.$$
- With this equation
$$\begin{aligned} A^T P + PA &= \int_0^\infty \left(A^T e^{A^T\tau} Q e^{A\tau} + e^{A^T\tau} Q e^{A\tau} A\right) d\tau \\ &= \int_0^\infty \tfrac{d}{d\tau}\left(e^{A^T\tau} Q e^{A\tau}\right) d\tau = e^{A^T t} Q e^{At}\Big|_0^\infty \\ &= \left(\lim_{t\to\infty} e^{A^T t} Q e^{At}\right) - e^{A^T 0} Q e^{A0} = -Q. \end{aligned}$$

- P is symmetric since $(Q = Q^T)$
$$\begin{aligned} P^T &= \int_0^\infty \left(e^{A^T\tau} Q e^{A\tau}\right)^T d\tau \\ &= \int_0^\infty e^{A^T\tau} Q e^{A\tau} d\tau = P. \end{aligned}$$
- $P \in \mathcal{S}^n_{>0}$: Let $z \in \mathbb{R}^n$ and consider
$$z^T P z = \int_0^\infty z^T e^{A^T\tau} Q e^{A\tau} z \, d\tau.$$
  - If $z \neq 0$ then $x(\tau) = e^{A\tau} z \neq 0$ and, since $Q \in \mathcal{S}^n_{>0}$ implies
$$z^T P z = \int_0^\infty x(\tau)^T Q x(\tau) d\tau > 0$$
  - If $z = 0$ then $x(\tau) = 0$
- (Uniqueness of P can be shown by contradiction)

Linearization (Local exponential stability)

Consider:

$$\dot{x} = f(x), \qquad f(0) = 0, \quad f \text{ cont. differentiable}$$

Define (Jacobian evaluated at the origin):

$$A = \left[\frac{\partial f(x)}{\partial x}\right]_{x=0} \qquad \text{(and define } f_1(x) = f(x) - Ax\text{)}$$

Note that

$$\lim_{|x|\to 0} \frac{|f_1(x)|}{|x|} = \lim_{|x|\to 0} \frac{|f(x) - Ax|}{|x|} = 0,$$

(which can be concluded from L'Hôpital's rule or the Taylor approximation)

Linearization of $\dot{x} = f(x)$ at $x = 0$:

$$\dot{z}(t) = Az(t)$$

Theorem

Consider $\dot{x} = f(x)$ (f cont. differentiable) and its linearization $\dot{z} = Az$. If the origin $z^e = 0$ of $\dot{z} = Az$ is globally exponentially stable then the origin $x^e = 0$ of $\dot{x} = f(x)$ is locally exponentially stable.

Proof:

- Let the origin of $\dot{z} = Az$ be exp. stable
- Define $Q = I$. Then there exists $P \in \mathcal{S}^n_{>0}$ so that

$$A^T P + PA = -I$$

- Take $V(x) = x^T Px$. Then

$$\langle \nabla V(x), f(x)\rangle = \langle 2Px, Ax - f_1(x)\rangle = -x^T x + 2x^T P f_1(x)$$

- Choose $r > 0$ and $\rho < \frac{1}{2}$ such that

$$|f_1(x)| \leq \frac{\rho}{\lambda_{\max}}|x| \qquad \forall\, |x| \leq r$$

- Then, for all $|x| \leq r$,

$$\begin{aligned} |2x^T P f_1(x)| &\leq 2|Px|\,|f_1(x)| \\ &\leq 2\left(\lambda_{\max}|x|\right)\left(\frac{\rho}{\lambda_{\max}}|x|\right) = 2\rho x^T x. \end{aligned}$$

- Therefore, for $|x| \leq r$, (and $c = \frac{1-2\rho}{\lambda_{\max}} > 0$, $\rho < \frac{1}{2}$)

$$\begin{aligned} \langle \nabla V(x), f(x)\rangle &\leq -x^T x + 2\rho x^T x = -(1-2\rho)x^T x \\ &\leq -\frac{1-2\rho}{\lambda_{\max}} V(x) = -cV(x) \end{aligned}$$

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- Take $V(x) = x^T Px$. Then

$$\langle \nabla V(x), f(x)\rangle = \langle 2Px, Ax - f_1(x)\rangle = -x^T x + 2x^T P f_1(x)$$

- Choose $r > 0$ and $\rho < \frac{1}{2}$ such that

$$|f_1(x)| \leq \frac{\rho}{\lambda_{\max}}|x| \qquad \forall\, |x| \leq r$$

- Then, for all $|x| \leq r$,

$$\begin{aligned} |2x^T P f_1(x)| &\leq 2|Px|\,|f_1(x)| \\ &\leq 2\left(\lambda_{\max}|x|\right)\left(\frac{\rho}{\lambda_{\max}}|x|\right) = 2\rho x^T x. \end{aligned}$$

- Therefore, for $|x| \leq r$, (and $c = \frac{1-2\rho}{\lambda_{\max}} > 0$, $\rho < \frac{1}{2}$)

$$\begin{aligned} \langle \nabla V(x), f(x)\rangle &\leq -x^T x + 2\rho x^T x = -(1-2\rho)x^T x \\ &\leq -\frac{1-2\rho}{\lambda_{\max}} V(x) = -cV(x) \end{aligned}$$

Linearization (Local exponential stability)

Consider:

$$\dot{x} = f(x), \qquad f(0) = 0, \quad f \text{ cont. differentiable}$$

Define (Jacobian evaluated at the origin):

$$A = \left[\frac{\partial f(x)}{\partial x}\right]_{x=0} \qquad \text{(and define } f_1(x) = f(x) - Ax\text{)}$$

Note that

$$\lim_{|x|\to 0} \frac{|f_1(x)|}{|x|} = \lim_{|x|\to 0} \frac{|f(x) - Ax|}{|x|} = 0,$$

(which can be concluded from L'Hôpital's rule or the Taylor approximation)

Linearization of $\dot{x} = f(x)$ at $x = 0$:

$$\dot{z}(t) = Az(t)$$

Theorem

Consider $\dot{x} = f(x)$ (f cont. differentiable) and its linearization $\dot{z} = Az$. If the origin $z^e = 0$ of $\dot{z} = Az$ is globally exponentially stable then the origin $x^e = 0$ of $\dot{x} = f(x)$ is locally exponentially stable.

Proof:

- Let the origin of $\dot{z} = Az$ be exp. stable
- Define $Q = I$. Then there exists $P \in \mathcal{S}^n_{>0}$ so that

$$A^T P + PA = -I$$

- Take $V(x) = x^T Px$. Then

$$\langle \nabla V(x), f(x)\rangle = \langle 2Px, Ax - f_1(x)\rangle = -x^T x + 2x^T Pf_1(x)$$

- Choose $r > 0$ and $\rho < \frac{1}{2}$ such that

$$|f_1(x)| \leq \frac{\rho}{\lambda_{\max}}|x| \qquad \forall\, |x| \leq r$$

- Then, for all $|x| \leq r$,

$$\begin{aligned}|2x^T Pf_1(x)| &\leq 2|Px|\,|f_1(x)| \\ &\leq 2\,(\lambda_{\max}|x|)\left(\frac{\rho}{\lambda_{\max}}|x|\right) = 2\rho x^T x.\end{aligned}$$

- Therefore, for $|x| \leq r$, (and $c = \frac{1-2\rho}{\lambda_{\max}} > 0$, $\rho < \frac{1}{2}$)

$$\begin{aligned}\langle \nabla V(x), f(x)\rangle &\leq -x^T x + 2\rho x^T x = -(1-2\rho)x^T x \\ &\leq -\frac{1-2\rho}{\lambda_{\max}} V(x) = -cV(x)\end{aligned}$$

Linearization (Stability, Instability & Limitations)

Theorem (Local Exponential Stability)

Consider $\dot{x} = f(x)$ (f cont. differentiable) and its linearization $\dot{z} = Az$. If the origin $z^e = 0$ of $\dot{z} = Az$ is globally exponentially stable then the origin $x^e = 0$ of $\dot{x} = f(x)$ is locally exponentially stable.

Theorem (Instability)

Consider the nonlinear system $\dot{x} = f(x)$ (f cont. differentiable) and its linearization $\dot{z} = Az$. The equilibrium 0 is unstable for $\dot{x} = f(x)$ if A has at least one eigenvalue with positive real part.

Note that

- if all eigenvalues of A have non-positive real part but A has any eigenvalues with zero real part, then the linearization is inconclusive.
  - $\dot{x} = x^3$ (the origin is unstable)
  - $\dot{x} = -x^3$ (the origin is asymptotically stable)
  - $\dot{z} = 0 \cdot z$ (linearization)
- f needs to be continuously differentiable

The role of the Lyapunov equation: $A^T P + PA = -Q$
Candidate Lyapunov functions: $V(x) = x^T P x$
Time derivative with respect to $\dot{x} = Ax$:

$$\frac{d}{dt}V(x) = \langle \nabla 2V(x(t)), \dot{x}(t) \rangle = \langle \nabla 2Px, Ax \rangle = 2x^T PAx$$
$$= x^T PAx + (x^T PAx)^T = x^T PAx + x^T A^T Px$$
$$= x^T (A^T P + PA)x = x^T(-Q)x = -x^T Qx$$

$$\frac{d}{dt}V(x) = \frac{d}{dt}(x^T Px) = \dot{x}^T Px + x^T P\dot{x} = x^T(A^T P + PA)x$$

Linearization (Stability, Instability & Limitations)

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Linearization (Stability, Instability & Limitations)

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Linearization (Local Lyapunov functions)

Corollary

Consider $\dot{x} = f(x)$ *(*f *cont. differentiable) and its linearization* $\dot{z} = Az$ *with a locally/globally exponentially stable origin of the linear/nonlinear dynamics. Let* $P \in \mathcal{S}^n_{>0}$ *be the unique solution of the Lyapunov Equation*

$$A^T P + PA = -Q, \qquad (Q \in \mathcal{S}^n_{>0} \text{ arbitrary}).$$

Then $V(x) = x^T P x$ *is a local Lyapunov function of the nonlinear system* $\dot{x} = f(x)$*.*

Thus:

- If the origin is locally exponentially stable, it is straightforward to define a local Lyapunov function.

However:

- It is not trivial to obtain a (good) estimate of the region of attraction
- While $Q \in \mathcal{S}^n_{>0}$ can be selected arbitrarily, P (and thus $V(x)$) depends on Q. Thus a possible estimate of the region of attraction depends on P (and Q)

The role of the Lyapunov equation: $A^T P + PA = -Q$
Candidate Lyapunov functions: $V(x) = x^T P x$
Time derivative with respect to $\dot{x} = Ax$:

$$\begin{aligned}\tfrac{d}{dt}V(x) &= \langle \nabla 2V(x(t)), \dot{x}(t)\rangle = \langle \nabla 2Px, Ax\rangle = 2x^T PAx \\ &= x^T PAx + (x^T PAx)^T = x^T PAx + x^T A^T Px \\ &= x^T(A^T P + PA)x = x^T(-Q)x = -x^T Qx\end{aligned}$$

$$\tfrac{d}{dt}V(x) = \tfrac{d}{dt}(x^T Px) = \dot{x}^T Px + x^T P\dot{x} = x^T(A^T P + PA)x$$

Linearization (Example 1: The linearization is inconclusive)

- Consider the nonlinear system

$$\dot{x} = cx^3, \qquad c \in \mathbb{R}$$

- Consider candidate Lyapunov function

$$V(x) = \tfrac{1}{2}x^2$$

- which satisfies

$$\dot{V}(x) = \langle \nabla V(x), cx^2 \rangle = cx^4.$$

  Thus,

  - for $c < 0$, the origin of $\dot{x} = cx^3$ is asymptotically stable
  - for $c > 0$ the origin of $\dot{x} = cx^3$ is unstable

  However,

  - independent of c, the linearization around the origin is given by $\dot{z} = Az = 0 \cdot z$.

  Hence,

  - (since the real part of the eigenvalue of A is zero) the linearization is inconclusive.

Linearization (Example 2: Mass-Spring System with Hardening String)

- Hardening spring:

$$F_{sp} = k_0 y + k_1 y^3 = k_0 x_1 + k_1 x_1^3, \quad \text{with } k_0, k_1 > 0$$

- Dynamics in state space form ($c, m > 0$):

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{1}{m}\left(-k_0 x_1 - k_1 x_1^3 - c x_2\right).$$

- Linearization at $x^e = 0$:

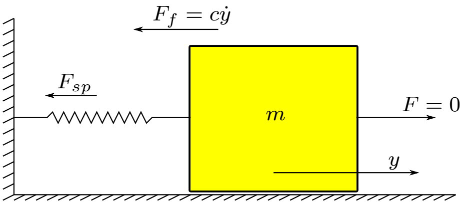
$$A = \left[\frac{\partial f(x)}{\partial x}\right]_{x=0} = \begin{bmatrix} 0 & 1 \\ -\frac{k_0}{m} - 3\frac{k_1}{m}x_1^2 & -\frac{c}{m} \end{bmatrix}_{x=0} = \begin{bmatrix} 0 & 1 \\ -\frac{k_0}{m} & -\frac{c}{m} \end{bmatrix}$$

- Eigenvalues of A:

$$0 = \det(\lambda I - A) = \lambda\left(\lambda + \frac{c}{m}\right) + \frac{k_0}{m} = \lambda^2 + \lambda\frac{c}{m} + \frac{k_0}{m}$$

i.e., $\lambda_{1,2} = -\frac{c}{2m} \pm \sqrt{\frac{c^2}{4m^2} - \frac{k_0}{m}}$.

- Identify three cases:
  - $k_0 = \frac{c^2}{4} \quad \rightsquigarrow \quad \mathrm{Re}(\lambda_{1,2}) < 0$
  - $k_0 < \frac{c^2}{4} \quad \rightsquigarrow \quad \mathrm{Re}(\lambda_{1,2}) < 0$
  - $k_0 > \frac{c^2}{4} \quad \rightsquigarrow \quad \mathrm{Re}(\lambda_{1,2}) < 0$
- Therefore,
  - the origin is globally exponentially stable for $\dot{z} = Az$
  - the origin is locally exponentially stable for $\dot{x} = f(x)$



Linearization (Example 3: Inverted Pendulum)

- Consider the pendulum:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{\ell}\sin(x_1+\pi) - \frac{k}{m}x_2.$$

(with origin shifted to the upright position)

- Matrix describing the linearized system:

$$A = \left[\frac{\partial f(x)}{\partial x}\right]_{x=0} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell}\cos(x_1+\pi) & -\frac{k}{m} \end{bmatrix}_{x=0}$$
$$= \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & -\frac{k}{m} \end{bmatrix}$$

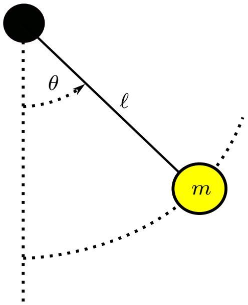
- The eigenvalues are defined through:

$$0 = \det(\lambda I - A) = \lambda\left(\lambda + \frac{k}{m}\right) - \frac{g}{\ell}$$
$$= \lambda^2 + \lambda\frac{k}{m} - \frac{g}{\ell}$$

so that

$$\lambda_{1,2} = -\frac{k}{2m} \pm \sqrt{\left(\frac{k}{2m}\right)^2 + \frac{g}{\ell}}$$

- One eigenvalue has
  - positive real part and
  - negative real part
- Thus,
  - the origin (upright position) of $\dot{z} = Az$ is unstable
  - the origin (upright position) of $\dot{x} = f(x)$ is unstable



Linearization (Example 3: Mass-Spring-Damper System)

- Consider the mass-spring damper system:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{1}{m}\left(-k_0 x_1 - k_1 x_1^3 - b x_2 |x_2|\right).$$

- The linearized system is described by

$$A = \left[\frac{\partial f(x)}{\partial x}\right]_{x=0} = \begin{bmatrix} 0 & 1 \\ -\frac{k_0}{m} - 3\frac{k_1}{m}x_1^2 & -2\frac{b}{m}x_2 \end{bmatrix}_{x=0}$$
$$= \begin{bmatrix} 0 & 1 \\ -\frac{k_0}{m} & 0 \end{bmatrix}.$$

- The eigenvalues are defined through

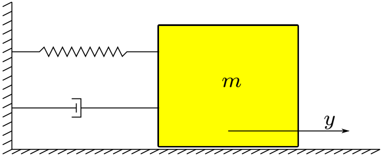
$$0 = \det(\lambda I - A) = \lambda^2 + \frac{k_0}{m}$$

which implies

$$\lambda = \pm j\sqrt{k_0/m}$$

.

- Since the eigenvalues are simple (i.e., multiplicity 1) A is diagonalizable. Since all the eigenvalues have zero real parts the origin of $\dot{z} = Az$ is stable
- Since the eigenvalues of A have zero real parts, the linearization tells us nothing about stability of the origin for $\dot{x} = f(x)$



Linear Time-Varying Systems

Linear time-invariant systems:

$$\dot{x}(t) = Ax(t)$$

Linear time-varying systems:

$$\dot{x}(t) = A(t)x(t)$$

Theorem (Stability of linear systems)

For the linear system $\dot{x} = Ax$, the origin is

1. *stable if and only if the eigenvalues of A have negative or zero real parts and all the Jordan blocks corresponding to eigenvalues with zero real parts are 1×1;*
2. *unstable if and only if at least one eigenvalue of A has a positive real part or zero real part with the corresponding Jordan block larger than 1×1;*
3. *exponentially stable if and only if all the eigenvalues of A have strictly negative real parts.*

Example

The matrix

$$A(t) = \begin{bmatrix} -1 + 1.5\cos^2(t) & 1 - 1.5\sin(t)\cos(t) \\ -1 - 1.5\sin(t)\cos(t) & -1 + 1.5\sin^2(t) \end{bmatrix}$$

has eigenvalues at

$$\lambda_{1,2} = -0.25 \pm j0.25\sqrt{7} \qquad \forall\, t \in \mathbb{R}_{\geq 0}$$

However, the solution of $\dot{x}(t) = A(t)x(t)$ is given by

$$x(t) = \begin{bmatrix} e^{0.5t}\cos(t) & e^{-t}\sin(t) \\ -e^{0.5t}\sin(t) & e^{-t}\cos(t) \end{bmatrix} x(0)$$

which clearly has a component that exponentially diverges from zero.

$\rightsquigarrow$ This result is not applicable to time-varying systems!

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Linear Time-Varying Systems, 2

- Time-invariant results relied on

$$\lim_{|x|\to 0} \frac{|f(x) - Ax|}{|x|} = 0$$

- However,

$$A(t) = \left[\frac{\partial f(t,x)}{\partial x}\right]_{x=0},$$

does not (necessarily) imply

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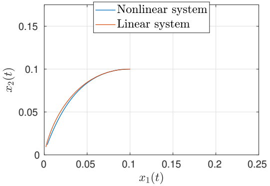
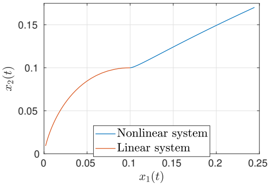


$t \in [0, 4]$



$t \in [10, 14]$

Example

Consider the time-varying system:

$$\dot{x} = f(t,x) = \begin{bmatrix} -x_1 + tx_2^2 \\ x_1 - x_2 \end{bmatrix}$$

with

$$\left[\frac{\partial f(t,x)}{\partial x}\right]_{x=0} x = A(t)x = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} x.$$

We see that

$$\lim_{|x|\to 0} \sup_{t\geq 0} \frac{|f(t,x) - A(t)x|}{|x|} \geq \lim_{|x_2|\to 0} \sup_{t\geq 0} \frac{|tx_2^2|}{|x_2|} \geq \lim_{x_2\to 0} \frac{|\frac{1}{x_2} x_2^2|}{|x_2|} = 1,$$

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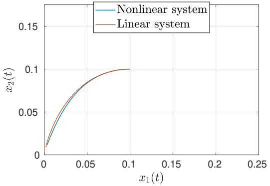

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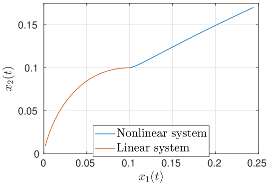

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We see that
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$$\geq \lim_{x_2\to 0} \frac{|\frac{1}{x_2}x_2^2|}{|x_2|} = 1,$$

Linear Time-Varying Systems, 3

Theorem

Consider $\dot{x} = f(t, x)$ *(*f *cont. differentiable) and suppose that* $f(t, 0) = 0$ *for all* $t \geq t_0$*.*
Assume that

$$\lim_{|x|\to 0} \sup_{t\geq 0} \frac{|f(t,x) - A(t)x|}{|x|} = 0.$$

holds and that

$$A(t) = \left[\frac{\partial f(t,x)}{\partial x}\right]_{x=0},$$

is bounded.
Then, if the origin is an exponentially stable equilibrium for $\dot{z}(t) = A(t)z(t)$ *then it is also an exponentially stable equilibrium of* $\dot{x} = f(t, x)$*.*

Numerical Calculation of Lyapunov Functions (Introduction)

Recall: Linear systems & Quadratic Lyapunov functions

$$\dot{x} = Ax, \qquad V(x) = x^T P x$$

Now, consider:

$$\dot{x} = f(x), \qquad f : \mathbb{R}^n \to \mathbb{R}^n \text{ polynomial}$$

A Lyapunov function

- is positive definite, i.e., $V(x) \geq 0$
- decreases along solutions, i.e., $\langle \nabla V(x), f(x) \rangle \leq 0$

Consider $W : \mathbb{R}^m \to \mathbb{R}$

- How can we validate if W is positive definite?
- If $W(z) = |Hz|^2 = z^T H^T H z$, then $W(z) \geq 0$.
- For $P \in \mathcal{S}^m_{\geq 0}$ there exists $H \in \mathbb{R}^{m \times m}$, $P = H^T H$

Goal: Construct Lyapunov functions of the form $V(x) = z(x)^T P z(x)$, $P \in \mathcal{S}^m_{\geq 0}$ where

- $V(x) = W(z(x))$, $W(z) = z^T P z$
- $z : \mathbb{R}^n \to \mathbb{R}^m$, $m \in \mathbb{N}$, denotes monomial functions

$$z_j(x) = \prod_{i=1}^n x_i^{j_i}$$

for $j_i \in \mathbb{N}$, for all $i \in \{1, \ldots, n\}$ for all $j \in \{1, \ldots, m\}$.

For example:

- Monomials of degree less than 3; $z : \mathbb{R}^2 \to \mathbb{R}^5$,

$$z(x) \doteq [\ x_1, x_2, x_1^2, x_2^2, x_1 x_2\ ]^T$$

- Monomials of degree less than 4; $y : \mathbb{R}^2 \to \mathbb{R}^9$,

$$y(x) \doteq [\ x_1, x_2, x_1^2, x_2^2, x_1 x_2, x_1^3, x_2^3, x_1^2 x_2, x_1 x_2^2\ ]^T$$

Theorem

Consider $\dot{x} = f(x)$ (f, polynomial, $f(0) = 0$), a domain $\mathcal{D} \subset \mathbb{R}^n$ and a function $\kappa : \mathbb{R}^n \to \mathbb{R}$ such that

$$\kappa(x) \leq 0 \ \forall\, x \in \mathcal{D} \quad \text{and} \quad \kappa(x) > 0 \ x \in \mathbb{R}^n \backslash \mathcal{D}.$$

Suppose we have a cont. differentiable fcn. $V : \mathbb{R}^n \to \mathbb{R}$, $\alpha_1, \rho \in \mathcal{K}_\infty$, and $\delta_1, \delta_2 : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfying

$$\alpha_1(|x|) - \delta_1(x)\kappa(x) \leq V(x) \quad \forall x \in \mathbb{R}^n$$

$$\langle \nabla V(x), f(x) \rangle \leq -\rho(|x|) + \delta_2(x)\kappa(x) \quad \forall x \in \mathbb{R}^n$$

Then the origin is locally asymptotically stable.
If $\mathcal{D} = \mathbb{R}^n$, then the origin is globally asymptotically stable.

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Now, consider:

$$\dot{x} = f(x), \qquad f : \mathbb{R}^n \to \mathbb{R}^n \text{ polynomial}$$

A Lyapunov function

- is positive definite, i.e., $V(x) \geq 0$
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Consider $W : \mathbb{R}^m \to \mathbb{R}$

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Goal: Construct Lyapunov functions of the form $V(x) = z(x)^T P z(x)$, $P \in \mathcal{S}^m_{\geq 0}$ where

- $V(x) = W(z(x))$, $W(z) = z^T P z$
- $z : \mathbb{R}^n \to \mathbb{R}^m$, $m \in \mathbb{N}$, denotes monomial functions

$$z_j(x) = \prod_{i=1}^n x_i^{j_i}$$

for $j_i \in \mathbb{N}$, for all $i \in \{1, \ldots, n\}$ for all $j \in \{1, \ldots, m\}$.

For example:

- Monomials of degree less than 3; $z : \mathbb{R}^2 \to \mathbb{R}^5$,

$$z(x) \doteq [\; x_1, x_2, x_1^2, x_2^2, x_1 x_2 \;]^T$$

- Monomials of degree less than 4; $y : \mathbb{R}^2 \to \mathbb{R}^9$,

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Theorem

Consider $\dot{x} = f(x)$ (f, polynomial, $f(0) = 0$), a domain $\mathcal{D} \subset \mathbb{R}^n$ and a function $\kappa : \mathbb{R}^n \to \mathbb{R}$ such that

$$\kappa(x) \leq 0 \; \forall \, x \in \mathcal{D} \quad \text{and} \quad \kappa(x) > 0 \; x \in \mathbb{R}^n \backslash \mathcal{D}.$$

Suppose we have a cont. differentiable fcn. $V : \mathbb{R}^n \to \mathbb{R}$, $\alpha_1, \rho \in \mathcal{K}_\infty$, and $\delta_1, \delta_2 : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfying

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Numerical Calculation of Lyapunov Functions (Introduction)

Recall: Linear systems & Quadratic Lyapunov functions

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Num. Calc. of Lyapunov fcns (Linear Matrix Inequalities and Semidefinite Programming)

Consider $\dot{x} = Ax$:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Consider the conditions:

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Define: (known functions/parameters)

- $\kappa(x) = 0$ ($\mathcal{D} = \mathbb{R}^n$, global results)
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Candidate functions: (unknown functions/parameters)

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Simplification:

$$-x^T P x + \varepsilon x^T x \leq 0, \qquad -x^T Q x + \varepsilon x^T x \leq 0$$
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Missing condition:

$$-x^T Q x = \langle \nabla V(x), f(x) \rangle = x^T (A^T P + PA) x$$

This implies

$$-q_{11}x_1^2 - q_{22}x_2^2 - 2q_{12}x_1x_2 =$$
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Conclusions:

- If the semidefinite program has a solution, then the origin of the linear system is globally exponentially stable
- Moreover, $V(x) = x^T P x$ is a Lyapunov function
- For the given example

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \quad \rightsquigarrow \quad P = \begin{bmatrix} 6.65 & 1.95 \\ 1.95 & 4.76 \end{bmatrix}$$

The optimization problem

- is convex
- can be solved efficiently

In Matlab (external toolboxes)

- CVX
- SOSTOOLS
- YALMIP

Note that, the unknown Q is not necessary:

$$\begin{aligned} \min_{P\in\mathcal{S}^2} \quad & 1 \\ \text{subject to} \quad & 0 \geq -P + \varepsilon I \\ & 0 \geq (A^T P + PA) + \varepsilon I \end{aligned}$$

Global Lyapunov functions for polynomial systems (1)

Consider the nonlinear system:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - x_2 + cx_1^3, \qquad c = -\tfrac{1}{4}$$

Candidate Lyapunov function:

$$V(x) = W(z(x)) = z(x)Pz(x)$$
$$z(x) \doteq \left[\; x_1, x_2, x_1^2, x_2^2, x_1 x_2 \;\right]^T$$
$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} & p_{15} \\ p_{12} & p_{22} & p_{23} & p_{24} & p_{25} \\ p_{13} & p_{23} & p_{33} & p_{34} & p_{35} \\ p_{14} & p_{24} & p_{34} & p_{44} & p_{45} \\ p_{15} & p_{25} & p_{35} & p_{45} & p_{55} \end{bmatrix}$$

Define: ($\varepsilon > 0$)

$$\kappa(x) = 0, \quad \alpha_1(|x|) = \rho(|x|) = \varepsilon x^T x,$$

Condition 1: ($\alpha_1(|x|) \leq V(x)$)

$$-P + \begin{bmatrix} \varepsilon I & 0 \\ 0 & 0 \end{bmatrix} \leq 0$$

Condition 2: ($\langle \nabla V(x), f(x) \rangle = z^T Q z \leq -\varepsilon x^T x$)

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Global Lyapunov functions for polynomial systems (Tedious Calculations 1)

$$
\begin{aligned}
V(x) &= p_{11}x_1^2 + 2p_{13}x_1^3 + p_{22}x_2^2 + 2p_{24}x_2^3 + p_{33}x_1^4 + p_{44}x_2^4 \\
&\quad + (2p_{14} + 2p_{25})x_1x_2^2 + (2p_{15} + 2p_{23})x_1^2x_2 + 2p_{35}x_1^3x_2 \\
&\quad + 2p_{45}x_1x_2^3 + (2p_{34} + p_{55})x_1^2x_2^2 + 2p_{12}x_1x_2, \\
\nabla_{x_1}V(x) &= 2p_{11}x_1 + 2p_{12}x_2 + 6p_{13}x_1^2 + (2p_{14} + 2p_{25})x_2^2 + 4p_{33}x_1^3 + 2p_{45}x_2^3 \\
&\quad + 6p_{35}x_1^2x_2 + (4p_{34} + 2p_{55})x_1x_2^2 + (4p_{15} + 4p_{23})x_1x_2, \\
\nabla_{x_2}V(x) &= 2p_{12}x_1 + 2p_{22}x_2 + (2p_{15} + 2p_{23})x_1^2 + 6p_{24}x_2^2 + 2p_{35}x_1^3 \\
&\quad + 4p_{44}x_2^3 + (4p_{34} + 2p_{55})x_1^2x_2 + 6p_{45}x_1x_2^2 + (4p_{14} + 4p_{25})x_1x_2,
\end{aligned}
$$

$$
\begin{aligned}
\langle \nabla V(x), f(x) \rangle &= (2p_{12} - 2p_{22})x_2^2 - 2p_{12}x_1^2 + (2cp_{12} - 2p_{35})x_1^4 \\
&\quad + (-2p_{15} - 2p_{23})x_1^3 + (2cp_{15} + 2cp_{23})x_1^5 + (2p_{14} - 6p_{24} + 2p_{25})x_2^3 \\
&\quad + 2cp_{35}x_1^6 + (-4p_{44} + 2p_{45})x_2^4 + (6p_{13} - 4p_{14} - 2p_{15} - 2p_{23} - 4p_{25})x_1^2x_2 \\
&\quad + (-4p_{14} + 4p_{15} + 4p_{23} - 6p_{24} - 4p_{25})x_1x_2^2 + (4cp_{14} + 4cp_{25})x_1^4x_2 \\
&\quad + (2cp_{22} + 4p_{33} - 4p_{34} - 2p_{35} - 2p_{55})x_1^3x_2 + (4cp_{34} + 2cp_{55})x_1^5x_2 \\
&\quad + (4p_{34} - 4p_{44} - 6p_{45} + 2p_{55})x_1x_2^3 + 6cp_{24}x_1^3x_2^2 \\
&\quad + (-4p_{34} + 6p_{35} - 6p_{45} - 2p_{55})x_1^2x_2^2 + 4cp_{44}x_1^3x_2^3 + 6cp_{45}x_1^4x_2^2 \\
&\quad + (2p_{11} - 2p_{12} - 2p_{22})x_1x_2.
\end{aligned}
$$

⇝ NOTE THAT $\langle \nabla V(x), f(x) \rangle$ has terms up to order 6.

Global Lyapunov functions for polynomial systems (2)

Consider the nonlinear system:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - x_2 + cx_1^3, \qquad c = -\tfrac{1}{4}$$

Candidate Lyapunov function:

$$V(x) = W(z(x)) = z(x)Pz(x)$$
$$z(x) \doteq \left[\; x_1, x_2, x_1^2, x_2^2, x_1x_2 \;\right]^T$$
$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} & p_{15} \\ p_{12} & p_{22} & p_{23} & p_{24} & p_{25} \\ p_{13} & p_{23} & p_{33} & p_{34} & p_{35} \\ p_{14} & p_{24} & p_{34} & p_{44} & p_{45} \\ p_{15} & p_{25} & p_{35} & p_{45} & p_{55} \end{bmatrix}$$

Define: $(\varepsilon > 0)$

$$\kappa(x) = 0, \quad \alpha_1(|x|) = \rho(|x|) = \varepsilon x^T x,$$

Condition 1: $(\alpha_1(|x|) \leq V(x))$

$$-P + \begin{bmatrix} \varepsilon I & 0 \\ 0 & 0 \end{bmatrix} \leq 0$$

Condition 2: $(\langle \nabla V(x), f(x)\rangle = z^T Q z \leq -\varepsilon x^T x)$

$\langle \nabla V(x), f(x)\rangle$ contains terms of degree 6 and thus can be written as

$$\langle \nabla V(x), f(x)\rangle = -y(x)^T Q y(x)$$
$$y(x) \doteq \left[\; x_1, x_2, x_1^2, x_2^2, x_1x_2, x_1^3, x_2^3, x_1^2x_2, x_1x_2^2 \;\right]^T$$
$$Q = \begin{bmatrix} q_{11} & \cdots & q_{19} \\ \vdots & \ddots & \vdots \\ q_{19} & \cdots & q_{99} \end{bmatrix} \in \mathcal{S}^9$$

Condition 2 is of the form

$$-Q + \begin{bmatrix} \varepsilon I & 0 \\ 0 & 0 \end{bmatrix} \leq 0$$

Expand $y^T Q y$:

$$\begin{aligned} &q_{11}x_1^2 + 2q_{13}x_1^3 + (2q_{16} + q_{33})x_1^4 + q_{22}x_2^2 + 2q_{24}x_2^3 + (2q_{27} + q_{44})x_2^4 \\ &+ 2q_{36}x_1^5 + 2q_{47}x_2^5 + q_{66}x_1^6 + q_{77}x_2^6 + (2q_{14} + 2q_{25})x_1x_2^2 + (2q_{15} + 2q_{23})x_1^2x_2 \\ &+ (2q_{17} + 2q_{29} + 2q_{45})x_1x_2^3 + (2q_{18} + 2q_{26} + 2q_{35})x_1^3x_2 + (2q_{38} + 2q_{56})x_1^4x_2 \\ &+ (2q_{49} + 2q_{57})x_1x_2^4 + 2q_{68}x_1^5x_2 + 2q_{79}x_1x_2^5 + (2q_{19} + 2q_{28} + 2q_{34} + q_{55})x_1^2x_2^2 \\ &+ (2q_{39} + 2q_{46} + 2q_{58})x_1^3x_2^2 + (2q_{37} + 2q_{48} + 2q_{59})x_1^2x_2^3 \\ &+ (2q_{67} + 2q_{89})x_1^3x_2^3 + (2q_{69} + q_{88})x_1^4x_2^2 + (2q_{78} + q_{99})x_1^2x_2^4 + 2q_{12}x_1x_2 \end{aligned}$$

Global Lyapunov functions for polynomial systems (Tedious Calculations 2)

x_1^2	$-2p_{12}$	$= -q_{11}$
x_1^3	$-2p_{15} - 2p_{23}$	$= -2q_{13}$
x_1^4	$2cp_{12} - 2p_{35}$	$= -(2q_{16} + q_{33})$
x_1^5	$2cp_{15} + 2cp_{23}$	$= -2q_{36}$
x_1^6	$2cp_{35}$	$= -q_{66}$
x_2^2	$2p_{12} - 2p_{22}$	$= -q_{22}$
x_2^3	$2p_{14} - 6p_{24} + 2p_{25}$	$= -2q_{24}$
x_2^4	$-4p_{44} + 2p_{45}$	$= -(2q_{27} + q_{44})$
x_2^5	0	$= -2q_{47}$
x_2^6	0	$= -q_{77}$
x_1x_2	$2p_{11} - 2p_{12} - 2p_{22}$	$= -2q_{12}$
$x_1x_2^2$	$-4p_{14} + 4p_{15} + 4p_{23} - 6p_{24} - 4p_{25}$	$= -(2q_{14} + 2q_{25})$
$x_1x_2^3$	$4p_{34} - 4p_{44} - 6p_{45} + 2p_{55}$	$= -(2q_{17} + 2q_{29} + 2q_{45})$
$x_1x_2^4$	0	$= -(2q_{49} + 2q_{57})$
$x_1x_2^5$	0	$= -2q_{79}$
$x_1^2x_2$	$6p_{13} - 4p_{14} - 2p_{15} - 2p_{23} - 4p_{25}$	$= -(2q_{15} + 2q_{23})$
$x_1^2x_2^2$	$-4p_{34} + 6p_{35} - 6p_{45} - 2p_{55}$	$= -(2q_{19} + 2q_{28} + 2q_{34} + q_{55})$
$x_1^2x_2^3$	0	$= -(2q_{37} + 2q_{48} + 2q_{59})$
$x_1^2x_2^4$	0	$= -(2q_{78} + q_{99})$
$x_1^3x_2$	$2cp_{22} + 4p_{33} - 4p_{34} - 2p_{35} - 2p_{55}$	$= -(2q_{18} + 2q_{26} + 2q_{35})$
$x_1^3x_2^2$	$6cp_{24}$	$= -(2q_{39} + 2q_{46} + 2q_{58})$
$x_1^3x_2^3$	$4cp_{44}$	$= -(2q_{67} + 2q_{89})$
$x_1^4x_2$	$4cp_{14} + 4cp_{25}$	$= -(2q_{38} + 2q_{56})$
$x_1^4x_2^2$	$6cp_{45}$	$= -(2q_{69} + q_{88})$
$x_1^5x_2$	$4cp_{34} + 2cp_{55}$	$= -2q_{68}$

Global Lyapunov functions for polynomial systems (Summary)

Consider the nonlinear system:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - x_2 + cx_1^3, \qquad c = -\tfrac{1}{4}$$

Optimization problem:

$$\min_{P\in\mathcal{S}^5,\ Q\in\mathcal{S}^9} \quad 1$$

subject to Linear Equality Constraints

$$0 \geq -P + \begin{bmatrix} \varepsilon I & 0 \\ 0 & 0 \end{bmatrix}$$
$$0 \geq -Q + \begin{bmatrix} \varepsilon I & 0 \\ 0 & 0 \end{bmatrix}$$

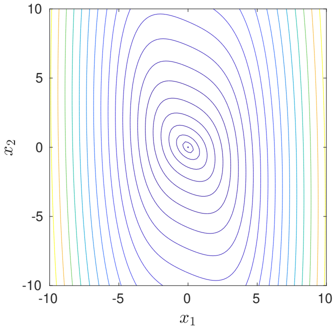
Summary:

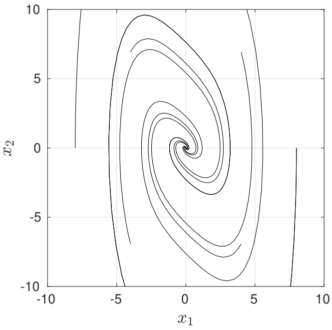
- If the semidefinite program is feasible, then the origin is globally asymptotically stable.
- Moreover, $V(x) = z(x)^T P z(x)$ is a Lyapunov function.

Here ($\varepsilon = 0.1$, $c = -\frac{1}{4}$):

$$P = \begin{bmatrix} 7.87 & 3.27 & 0.00 & 0.00 & 0.00 \\ 3.27 & 7.59 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.97 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \end{bmatrix}$$

$$V(x) = 7.87x_1^2 + 7.59x_2^2 + 6.54x_1x_2 + 0.96x_1^4.$$





Local Lyapunov functions for polynomial systems, 1

Consider the nonlinear system:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 - x_2 + c x_1^3, \qquad c = \frac{1}{4}$$

Three equilibria $x_1 \in \{0, \pm 2\}$, $x_2 = 0$ (i.e., the origin can't be globally asym. stable.)

Theorem

Consider $\dot{x} = f(x)$ (f, polynomial, $f(0) = 0$), a domain $\mathcal{D} \subset \mathbb{R}^n$ and a function $\kappa : \mathbb{R}^n \to \mathbb{R}$ such that

$$\kappa(x) \leq 0 \; \forall \, x \in \mathcal{D} \quad \text{and} \quad \kappa(x) > 0 \; x \in \mathbb{R}^n \backslash \mathcal{D}.$$

Suppose we have a cont. differentiable fcn. $V : \mathbb{R}^n \to \mathbb{R}$, $\alpha_1, \rho \in \mathcal{K}_\infty$, and $\delta_1, \delta_2 : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfying

$$\alpha_1(|x|) - \delta_1(x)\kappa(x) \leq V(x) \quad \forall x \in \mathbb{R}^n$$

$$\langle \nabla V(x), f(x) \rangle \leq -\rho(|x|) + \delta_2(x)\kappa(x) \quad \forall x \in \mathbb{R}^n$$

Then the origin is locally asymptotically stable.
If $\mathcal{D} = \mathbb{R}^n$, then the origin is globally asymptotically stable.

Consider $\mathcal{D} = \mathcal{B}_1(0) = \{x \in \mathbb{R}^n : |x| < 1\}$ and define

$$\kappa(x) = x^T x - 1$$

Unknown functions δ_1, δ_2:

$$\delta_1(x) = z(x)^T D_{sm} z(x) \quad \text{and} \quad \delta_2(x) = z(x)^T E_{sm} z(x)$$

where

$$D_{sm} = \begin{bmatrix} d_{11} & \cdots & d_{15} \\ \vdots & \ddots & \vdots \\ d_{15} & \cdots & d_{55} \end{bmatrix}, \quad E_{sm} = \begin{bmatrix} e_{11} & \cdots & e_{15} \\ \vdots & \ddots & \vdots \\ e_{15} & \cdots & e_{55} \end{bmatrix}$$

Calculate product $\delta_1(x)\kappa(x)$:

$$\begin{aligned} \delta_1(x)\kappa(x) &= z(x)^T D_{sm} z(x) \cdot (x^T x - 1) \\ &= z(x)^T D_{sm} z(x) x_1^2 + z(x)^T D_{sm} z(x) x_2^2 - z(x)^T D_{sm} z(x) \\ &= y(x)^T D_1 y(x) + y(x)^T D_2 y(x) - y(x)^T D_3 y(x) \\ &= y(x)^T D_{la} y(x) \end{aligned}$$

Here $D_1, D_2, D_3, D_{la} \in \mathcal{S}^9$. For example:

$$D_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d_{11} & 0 & d_{12} & d_{13} & 0 & d_{15} & d_{14} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d_{12} & 0 & d_{22} & d_{23} & 0 & d_{25} & d_{24} \\ 0 & 0 & d_{13} & 0 & d_{23} & d_{33} & 0 & d_{35} & d_{34} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d_{15} & 0 & d_{25} & d_{35} & 0 & d_{55} & d_{45} \\ 0 & 0 & d_{14} & 0 & d_{24} & d_{34} & 0 & d_{45} & d_{44} \end{bmatrix}$$

Local Lyapunov functions for polynomial systems, 1

Consider the nonlinear system:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 - x_2 + c x_1^3, \qquad c = \frac{1}{4}$$

Three equilibria $x_1 \in \{0, \pm 2\}$, $x_2 = 0$ (i.e., the origin can't be globally asym. stable.)

Theorem

Consider $\dot{x} = f(x)$ (f, polynomial, $f(0) = 0$), a domain $\mathcal{D} \subset \mathbb{R}^n$ and a function $\kappa : \mathbb{R}^n \to \mathbb{R}$ such that

$$\kappa(x) \leq 0 \; \forall \, x \in \mathcal{D} \quad \text{and} \quad \kappa(x) > 0 \; x \in \mathbb{R}^n \backslash \mathcal{D}.$$

Suppose we have a cont. differentiable fcn. $V : \mathbb{R}^n \to \mathbb{R}$, $\alpha_1, \rho \in \mathcal{K}_\infty$, and $\delta_1, \delta_2 : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfying

$$\alpha_1(|x|) - \delta_1(x)\kappa(x) \leq V(x) \qquad \forall x \in \mathbb{R}^n$$

$$\langle \nabla V(x), f(x) \rangle \leq -\rho(|x|) + \delta_2(x)\kappa(x) \qquad \forall x \in \mathbb{R}^n$$

Then the origin is locally asymptotically stable.
If $\mathcal{D} = \mathbb{R}^n$, then the origin is globally asymptotically stable.

Consider $\mathcal{D} = \mathcal{B}_1(0) = \{x \in \mathbb{R}^n : |x| < 1\}$ and define

$$\kappa(x) = x^T x - 1$$

Unknown functions δ_1, δ_2:

$$\delta_1(x) = z(x)^T D_{sm} z(x) \quad \text{and} \quad \delta_2(x) = z(x)^T E_{sm} z(x)$$

where

$$D_{sm} = \begin{bmatrix} d_{11} & \cdots & d_{15} \\ \vdots & \ddots & \vdots \\ d_{15} & \cdots & d_{55} \end{bmatrix}, \quad E_{sm} = \begin{bmatrix} e_{11} & \cdots & e_{15} \\ \vdots & \ddots & \vdots \\ e_{15} & \cdots & e_{55} \end{bmatrix}$$

Calculate product $\delta_1(x)\kappa(x)$:

$$\begin{aligned} \delta_1(x)\kappa(x) &= z(x)^T D_{sm} z(x) \cdot (x^T x - 1) \\ &= z(x)^T D_{sm} z(x) x_1^2 + z(x)^T D_{sm} z(x) x_2^2 - z(x)^T D_{sm} z(x) \\ &= y(x)^T D_1 y(x) + y(x)^T D_2 y(x) - y(x)^T D_3 y(x) \\ &= y(x)^T D_{la} y(x) \end{aligned}$$

Here $D_1, D_2, D_3, D_{la} \in \mathcal{S}^9$. For example:

$$D_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d_{11} & 0 & d_{12} & d_{13} & 0 & d_{15} & d_{14} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d_{12} & 0 & d_{22} & d_{23} & 0 & d_{25} & d_{24} \\ 0 & 0 & d_{13} & 0 & d_{23} & d_{33} & 0 & d_{35} & d_{34} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d_{15} & 0 & d_{25} & d_{35} & 0 & d_{55} & d_{45} \\ 0 & 0 & d_{14} & 0 & d_{24} & d_{34} & 0 & d_{45} & d_{44} \end{bmatrix}$$

Local Lyapunov functions for polynomial systems, 1

Consider the nonlinear system:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 - x_2 + cx_1^3, \qquad c = \frac{1}{4}$$

Three equilibria $x_1 \in \{0, \pm 2\}$, $x_2 = 0$ (i.e., the origin can't be globally asym. stable.)

Theorem

Consider $\dot{x} = f(x)$ (f, polynomial, $f(0) = 0$), a domain $\mathcal{D} \subset \mathbb{R}^n$ and a function $\kappa : \mathbb{R}^n \to \mathbb{R}$ such that

$$\kappa(x) \leq 0 \;\forall\, x \in \mathcal{D} \quad \text{and} \quad \kappa(x) > 0 \; x \in \mathbb{R}^n \backslash \mathcal{D}.$$

Suppose we have a cont. differentiable fcn. $V : \mathbb{R}^n \to \mathbb{R}$, $\alpha_1, \rho \in \mathcal{K}_\infty$, and $\delta_1, \delta_2 : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfying

$$\alpha_1(|x|) - \delta_1(x)\kappa(x) \leq V(x) \qquad \forall x \in \mathbb{R}^n$$

$$\langle \nabla V(x), f(x) \rangle \leq -\rho(|x|) + \delta_2(x)\kappa(x) \qquad \forall x \in \mathbb{R}^n$$

Then the origin is locally asymptotically stable.
If $\mathcal{D} = \mathbb{R}^n$, then the origin is globally asymptotically stable.

Consider $\mathcal{D} = \mathcal{B}_1(0) = \{x \in \mathbb{R}^n : |x| < 1\}$ and define

$$\kappa(x) = x^T x - 1$$

Unknown functions δ_1, δ_2:

$$\delta_1(x) = z(x)^T D_{sm} z(x) \quad \text{and} \quad \delta_2(x) = z(x)^T E_{sm} z(x)$$

where

$$D_{sm} = \begin{bmatrix} d_{11} & \cdots & d_{15} \\ \vdots & \ddots & \vdots \\ d_{15} & \cdots & d_{55} \end{bmatrix}, \quad E_{sm} = \begin{bmatrix} e_{11} & \cdots & e_{15} \\ \vdots & \ddots & \vdots \\ e_{15} & \cdots & e_{55} \end{bmatrix}$$

Calculate product $\delta_1(x)\kappa(x)$:

$$\begin{aligned} \delta_1(x)\kappa(x) &= z(x)^T D_{sm} z(x) \cdot (x^T x - 1) \\ &= z(x)^T D_{sm} z(x) x_1^2 + z(x)^T D_{sm} z(x) x_2^2 - z(x)^T D_{sm} z(x) \\ &= y(x)^T D_1 y(x) + y(x)^T D_2 y(x) - y(x)^T D_3 y(x) \\ &= y(x)^T D_{la} y(x) \end{aligned}$$

Here $D_1, D_2, D_3, D_{la} \in \mathcal{S}^9$. For example:

$$D_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d_{11} & 0 & d_{12} & d_{13} & 0 & d_{15} & d_{14} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d_{12} & 0 & d_{22} & d_{23} & 0 & d_{25} & d_{24} \\ 0 & 0 & d_{13} & 0 & d_{23} & d_{33} & 0 & d_{35} & d_{34} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d_{15} & 0 & d_{25} & d_{35} & 0 & d_{55} & d_{45} \\ 0 & 0 & d_{14} & 0 & d_{24} & d_{34} & 0 & d_{45} & d_{44} \end{bmatrix}$$

Local Lyapunov functions for polynomial systems, 1

Consider the nonlinear system:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 - x_2 + c x_1^3, \qquad c = \frac{1}{4}$$

Three equilibria $x_1 \in \{0, \pm 2\}$, $x_2 = 0$ (i.e., the origin can't be globally asym. stable.)

Theorem

Consider $\dot{x} = f(x)$ *(*f*, polynomial,* $f(0) = 0$*), a domain* $\mathcal{D} \subset \mathbb{R}^n$ *and a function* $\kappa : \mathbb{R}^n \to \mathbb{R}$ *such that*

$$\kappa(x) \leq 0 \; \forall \, x \in \mathcal{D} \quad \text{and} \quad \kappa(x) > 0 \; x \in \mathbb{R}^n \backslash \mathcal{D}.$$

Suppose we have a cont. differentiable fcn. $V : \mathbb{R}^n \to \mathbb{R}$, $\alpha_1, \rho \in \mathcal{K}_\infty$*, and* $\delta_1, \delta_2 : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *satisfying*

$$\alpha_1(|x|) - \delta_1(x)\kappa(x) \leq V(x) \qquad \forall x \in \mathbb{R}^n$$

$$\langle \nabla V(x), f(x) \rangle \leq -\rho(|x|) + \delta_2(x)\kappa(x) \qquad \forall x \in \mathbb{R}^n$$

Then the origin is locally asymptotically stable.
If $\mathcal{D} = \mathbb{R}^n$*, then the origin is globally asymptotically stable.*

Consider $\mathcal{D} = \mathcal{B}_1(0) = \{x \in \mathbb{R}^n : |x| < 1\}$ and define

$$\kappa(x) = x^T x - 1$$

Unknown functions δ_1, δ_2:

$$\delta_1(x) = z(x)^T D_{sm} z(x) \quad \text{and} \quad \delta_2(x) = z(x)^T E_{sm} z(x)$$

where

$$D_{sm} = \begin{bmatrix} d_{11} & \cdots & d_{15} \\ \vdots & \ddots & \vdots \\ d_{15} & \cdots & d_{55} \end{bmatrix}, \quad E_{sm} = \begin{bmatrix} e_{11} & \cdots & e_{15} \\ \vdots & \ddots & \vdots \\ e_{15} & \cdots & e_{55} \end{bmatrix}$$

Calculate product $\delta_1(x)\kappa(x)$:

$$\begin{aligned}
\delta_1(x)\kappa(x) &= z(x)^T D_{sm} z(x) \cdot (x^T x - 1) \\
&= z(x)^T D_{sm} z(x) x_1^2 + z(x)^T D_{sm} z(x) x_2^2 - z(x)^T D_{sm} z(x) \\
&= y(x)^T D_1 y(x) + y(x)^T D_2 y(x) - y(x)^T D_3 y(x) \\
&= y(x)^T D_{la} y(x)
\end{aligned}$$

Here $D_1, D_2, D_3, D_{la} \in \mathcal{S}^9$. For example:

$$D_1 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & d_{11} & 0 & d_{12} & d_{13} & 0 & d_{15} & d_{14} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & d_{12} & 0 & d_{22} & d_{23} & 0 & d_{25} & d_{24} \\
0 & 0 & d_{13} & 0 & d_{23} & d_{33} & 0 & d_{35} & d_{34} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & d_{15} & 0 & d_{25} & d_{35} & 0 & d_{55} & d_{45} \\
0 & 0 & d_{14} & 0 & d_{24} & d_{34} & 0 & d_{45} & d_{44}
\end{bmatrix}$$

Local Lyapunov functions for polynomial systems, 2

Consider the nonlinear system:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 - x_2 + c x_1^3, \qquad c = \tfrac{1}{4}$$

Three equilibria $x_1 \in \{0, \pm 2\}$, $x_2 = 0$ (i.e., the origin can't be globally asym. stable.)

Theorem

Consider $\dot{x} = f(x)$ (f, polynomial, $f(0) = 0$), a domain $\mathcal{D} \subset \mathbb{R}^n$ and a function $\kappa : \mathbb{R}^n \to \mathbb{R}$ such that

$$\kappa(x) \le 0 \; \forall\, x \in \mathcal{D} \quad \text{and} \quad \kappa(x) > 0 \; x \in \mathbb{R}^n \backslash \mathcal{D}.$$

Suppose we have a cont. differentiable fcn. $V : \mathbb{R}^n \to \mathbb{R}$, $\alpha_1, \rho \in \mathcal{K}_\infty$, and $\delta_1, \delta_2 : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ satisfying

$$\alpha_1(|x|) - \delta_1(x)\kappa(x) \le V(x) \qquad \forall x \in \mathbb{R}^n$$

$$\langle \nabla V(x), f(x) \rangle \le -\rho(|x|) + \delta_2(x)\kappa(x) \qquad \forall x \in \mathbb{R}^n$$

Then the origin is locally asymptotically stable.
If $\mathcal{D} = \mathbb{R}^n$, then the origin is globally asymptotically stable.

Consider $\mathcal{D} = \mathcal{B}_1(0) = \{x \in \mathbb{R}^n : |x| < 1\}$ and define

$$\kappa(x) = x^T x - 1$$

$D_{la}, E_{la} \in \mathcal{S}^9$ defined through unknowns $D_{sm}, E_{sm} \in \mathcal{S}^5$:

$$\delta_1(x) = z(x)^T D_{sm} z(x), \;\; \delta_1(x)\kappa(x) = y(x)^T D_{la} y(x)$$

$$\delta_2(x) = z(x)^T E_{sm} z(x), \;\; \delta_2(x)\kappa(x) = y(x)^T E_{la} y(x)$$

Corresponding feasibility problem:

$$\min_{\substack{P \in \mathcal{S}^5,\, Q \in \mathcal{S}^9 \\ D_{sm}, E_{sm} \in \mathcal{S}^5 \\ \hat{D}, \hat{E} \in \mathcal{S}^9}} \quad 1$$

subject to Linear equality constraints

$$0 \ge \begin{bmatrix} -P & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \varepsilon I & 0 \\ 0 & 0 \end{bmatrix} - \hat{D}$$

$$0 \ge -Q + \begin{bmatrix} \varepsilon I & 0 \\ 0 & 0 \end{bmatrix} - \hat{E}$$

$$0 \ge -D_{sm}$$

$$0 \ge -E_{sm}$$

$$0 = \hat{D} - D_{la}$$

$$0 = \hat{E} - E_{la}$$

Local Lyapunov functions for polynomial systems, 3

Consider the nonlinear system:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 - x_2 + cx_1^3, \qquad c = \frac{1}{4}$$

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Consider $\dot{x} = f(x)$ (f, polynomial, $f(0) = 0$), a domain $\mathcal{D} \subset \mathbb{R}^n$ and a function $\kappa : \mathbb{R}^n \to \mathbb{R}$ such that

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Suppose we have a cont. differentiable fcn. $V : \mathbb{R}^n \to \mathbb{R}$, $\alpha_1, \rho \in \mathcal{K}_\infty$, and $\delta_1, \delta_2 : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfying

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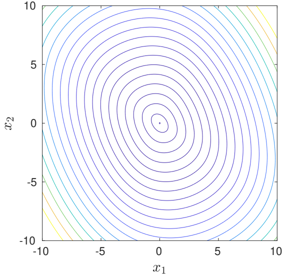
Then the origin is locally asymptotically stable.
If $\mathcal{D} = \mathbb{R}^n$, then the origin is globally asymptotically stable.

Consider $\mathcal{D} = \mathcal{B}_1(0) = \{x \in \mathbb{R}^n : |x| < 1\}$ and define

$$\kappa(x) = x^T x - 1$$

Here, $V(x) = z(x)^T P z(x)$:

$$P = \begin{bmatrix} 8.69 & 3.50 & 0 & 0 & 0 \\ 3.50 & 7.63 & 0 & 0 & 0 \\ 0 & 0 & 5.40 & 1.08 & 2.42 \\ 0 & 0 & 1.08 & 2.66 & 0.64 \\ 0 & 0 & 2.42 & 0.64 & 5.78 \end{bmatrix}$$



Remember:

- Feasibility implies local asymptotic stability
- Infeasibility does not imply that the origin is not asymptotically stable

Estimation of the region of attraction

Consider the nonlinear system:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 - x_2 + cx_1^3, \qquad c = \tfrac{1}{4}$$

Properties:

- $V(x)$ is positive definite and satisfies the decrease condition on $\mathcal{B}_1(0)$
- However, $\mathcal{B}_1(0)$ is not necessarily forward invariant

Theorem

Consider $\dot{x} = f(x)$ (f polynomial, $f(0) = 0$) a domain $\mathcal{D} \subset \mathbb{R}^n$ and a function $\kappa : \mathbb{R}^n \to \mathbb{R}$ such that $\kappa(x) \leq 0$ for all $x \in \mathcal{D}$ and $\kappa(x) > 0$ for all $x \in \mathbb{R}^n \backslash \mathcal{D}$. Additionally, let $V : \mathbb{R}^n \to \mathbb{R}$ be a Lyapunov function and let $k \in \mathbb{N}$, $\delta_3 : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and $c \in \mathbb{R}_{>0}$.

If $\quad (V(x) - c)|x|^{2k} - \delta_3(x)\kappa(x) \geq 0 \qquad x \in \mathbb{R}^n,$

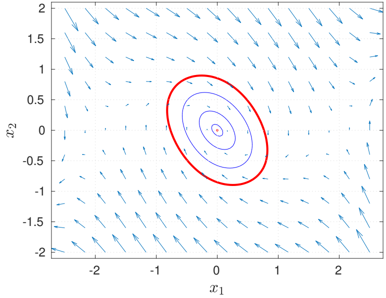
then the sublevel set $\{x \in \mathbb{R}^n : V(x) \leq c\} \subset \mathcal{D}$ is contained in the region of attraction.

Corresponding optimization problem:

$$\max_{\substack{c \in \mathbb{R} \\ \delta_3 : \mathbb{R}^n \to \mathbb{R}_{\geq 0}}} \quad c$$

subject to $\quad (V(x) - c)|x|^{2k} - \delta_3(x)\kappa(x) \geq 0 \qquad \forall\, x \in \mathbb{R}^n.$

With $k = 1$ and unknown polynomial δ_3 of order ≤ 4 optimal value $c^\star = 6.96$ is returned



Final comment:

- Estimate is quite conservative. The estimate can be improved by changing κ and by increasing k

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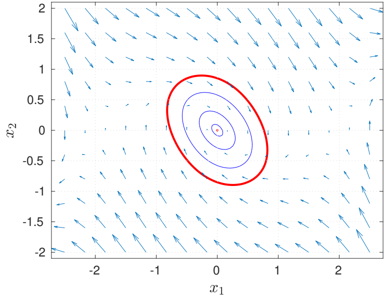

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- Estimate is quite conservative. The estimate can be improved by changing κ and by increasing k

Systems with Inputs

Consider:

$$\dot{x} = f(x,u), \quad (f\text{, cont. differentiable w.r.t. } x \text{ and } u)$$

Recall:

- An equilibrium pair (x^e, u^e) satisfies $f(x^e, u^e) = 0$
- Without loss of generality $f(0,0) = 0$ (due to coordinate transformation $z = x - x^e$, $v = u - u^e$)

Linearization:

$$A = \left[\frac{\partial f}{\partial x}(x,u)\right]_{(x,u)=0}, \qquad B = \left[\frac{\partial f}{\partial u}(x,u)\right]_{(x,u)=0}$$

Linear system with input:

$$\dot{x} = Ax + Bu, \qquad A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m}$$

The system is defined through the pair (A, B)
Solution (depending on $x(0)$ and u):

$$x(t; x_0, u) = x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

Output equation:

$$y = Cx + Du$$

Note that

- A linear system (with output) is unambiguously defined through (A, B) (or (A, B, C, D))
- (A, B) describes the system without output (or $x = y$)
- (A, C) describes output behavior without input
- The matrix D (direct feedthrough) is often not present

Example (Pendulum on cart; upright position)

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ \frac{-\bar{J}\bar{c}x_3 - \bar{J}\sin(x_2)x_4^2 - \bar{\gamma}\cos(x_2)x_4 + g\cos(x_2)\sin(x_2) + \bar{J}u}{\bar{M}\bar{J} - \cos^2(x_2)} \\ \frac{-\bar{M}\bar{\gamma}x_4 + \bar{M}g\sin(x_2) - \bar{c}\cos(x_2)x_3 - \cos(x_2)\sin(x_2)x_4^2 + \cos(x_2)u}{\bar{M}\bar{J} - \cos^2(x_2)} \end{bmatrix}$$

$$A = \left[\frac{\partial f}{\partial x}(x,u)\right]_{(0,0)} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{g}{\bar{M}\bar{J}-1} & -\frac{\bar{J}\bar{c}}{\bar{M}\bar{J}-1} & -\frac{\bar{\gamma}}{\bar{M}\bar{J}-1} \\ 0 & \frac{\bar{M}g}{\bar{M}\bar{J}-1} & -\frac{\bar{c}}{\bar{M}\bar{J}-1} & -\frac{\bar{M}\bar{\gamma}}{\bar{M}\bar{J}-1} \end{bmatrix}$$

$$B = \left[\frac{\partial f}{\partial u}(x,u)\right]_{(0,0)} = \begin{bmatrix} 0 \\ 0 \\ \frac{\bar{J}}{\bar{M}\bar{J}-1} \\ \frac{1}{\bar{M}\bar{J}-1} \end{bmatrix}$$

Systems with Inputs

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Linearization:

$$A = \left[\frac{\partial f}{\partial x}(x,u)\right]_{(x,u)=0}, \qquad B = \left[\frac{\partial f}{\partial u}(x,u)\right]_{(x,u)=0}$$

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Systems with Inputs (Controllability and Observability, 1)

Definition (Controllability)

Consider the linear system defined through (A, B). The linear system (or equivalently the pair (A, B)) is said to be controllable, if for all $x_1, x_2 \in \mathbb{R}^n$ there exists $T \in \mathbb{R}_{\geq 0}$ and $u : [0, T] \to \mathbb{R}^m$ such that

$$x_2 = e^{AT} x_1 + \int_0^T e^{A(T-\tau)} B u(\tau) d\tau.$$

Ability of a system to steer any initial state to a target state through an appropriate input $u : [0, T] \to \mathbb{R}^m$.

Definition (Observability)

Consider the linear system defined through (A, C). The linear system with output (or equivalently the pair (A, C)) is said to be observable, if for all $x_1, x_2 \in \mathbb{R}^n$, $x_1 \neq x_2$ there exists $T \in \mathbb{R}_{\geq 0}$ such that

$$Ce^{AT} x_2 \neq Ce^{AT} x_1.$$

Determines if $x(0)$ can be uniquely determined by measuring $y(t) = Cx(t)$ over a given time window $t \in [0, T]$.

Note that:

- Controllability and observability are independent of D.
- The triple (A, B, C) is called controllable and observable, if the pair (A, B) is controllable and the pair (A, C) is observable.
- Controllability and observability are independent concepts:

$$Ce^{AT} x_2 + C \int_0^T e^{A(T-\tau)} B u(\tau) d\tau$$
$$\neq Ce^{AT} x_1 + C \int_0^T e^{A(T-\tau)} B u(\tau) d\tau$$

Systems with Inputs (Controllability and Observability, 1)

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Consider the linear system defined through (A, C). The linear system with output (or equivalently the pair (A, C)) is said to be observable, if for all $x_1, x_2 \in \mathbb{R}^n$, $x_1 \neq x_2$ there exists $T \in \mathbb{R}_{\geq 0}$ such that

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Systems with Inputs (Controllability and Observability, 2)

Definition (Controllability)

Consider the linear system defined through (A, B). The linear system (or equivalently the pair (A, B)) is said to be controllable, if for all $x_1, x_2 \in \mathbb{R}^n$ there exists $T \in \mathbb{R}_{\geq 0}$ and $u : [0, T] \to \mathbb{R}^m$ such that

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Consider the linear system defined through (A, C). The linear system with output (or equivalently the pair (A, C)) is said to be observable, if for all $x_1, x_2 \in \mathbb{R}^n$, $x_1 \neq x_2$ there exists $T \in \mathbb{R}_{\geq 0}$ such that

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Determines if $x(0)$ can be uniquely determined by measuring $y(t) = Cx(t)$ over a given time window $t \in [0, T]$.

Theorem (Controllability, Kalman matrix)

Consider the linear system defined through the pair (A, B). The linear system (or equivalently the pair (A, B)) is controllable if and only if

$$\operatorname{rank}\left([B \; AB \; A^2B \; \cdots \; A^{n-1}B]\right) = n.$$

Theorem (Observability)

Consider the linear system defined through the pair (A, C). The linear system with output (or equivalently the pair (A, C)) is observable if and only if

$$\operatorname{rank}\left(\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}\right) = n.$$

- (A, B) controllable if and only if (A^T, B^T) observable
- (A, C) observable if and only if (A^T, C^T) controllable

Systems with Inputs (Controllability and Observability, 2)

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Systems with Inputs (Controllability and Observability, 2)

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Systems with Inputs (Controllability and Observability, 3)

Definition (Controllability)

Consider the linear system defined through (A, B). The linear system (or equivalently the pair (A, B)) is said to be controllable, if for all $x_1, x_2 \in \mathbb{R}^n$ there exists $T \in \mathbb{R}_{\geq 0}$ and $u : [0, T] \to \mathbb{R}^m$ such that

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Ability of a system to steer any initial state to a target state through an appropriate input $u : [0, T] \to \mathbb{R}^m$.

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Consider the linear system defined through (A, C). The linear system with output (or equivalently the pair (A, C)) is said to be observable, if for all $x_1, x_2 \in \mathbb{R}^n$, $x_1 \neq x_2$ there exists $T \in \mathbb{R}_{\geq 0}$ such that

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Determines if $x(0)$ can be uniquely determined by measuring $y(t) = Cx(t)$ over a given time window $t \in [0, T]$.

Theorem (Popov-Belevitch-Hautus (PBH) test)

The linear system defined through (A, B) is controllable if and only if

$$\operatorname{rank}\left(\begin{bmatrix} A - \lambda I & B \end{bmatrix}\right) = n \qquad \forall \lambda \in \mathbb{C}$$

Theorem (PBH test)

The linear system (or equivalently the pair (A, C)) is observable if and only if

$$\operatorname{rank}\left(\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}\right) = n \qquad \forall \lambda \in \mathbb{C}$$

Note that:

- The rank of a matrix needs to be considered with caution. Example

$$M_\varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon \end{bmatrix}, \qquad \varepsilon \neq 0$$

- Controllability/Observability is independent of the time interval $[0, T]$. In particular, T can be chosen arbitrarily small.

Systems with Inputs (Controllability and Observability, 3)

Definition (Controllability)

Consider the linear system defined through (A, B). The linear system (or equivalently the pair (A, B)) is said to be controllable, if for all $x_1, x_2 \in \mathbb{R}^n$ there exists $T \in \mathbb{R}_{\geq 0}$ and $u : [0, T] \to \mathbb{R}^m$ such that

$$x_2 = e^{AT} x_1 + \int_0^T e^{A(T-\tau)} B u(\tau) d\tau.$$

Ability of a system to steer any initial state to a target state through an appropriate input $u : [0, T] \to \mathbb{R}^m$.

Definition (Observability)

Consider the linear system defined through (A, C). The linear system with output (or equivalently the pair (A, C)) is said to be observable, if for all $x_1, x_2 \in \mathbb{R}^n$, $x_1 \neq x_2$ there exists $T \in \mathbb{R}_{\geq 0}$ such that

$$Ce^{AT} x_2 \neq Ce^{AT} x_1.$$

Determines if $x(0)$ can be uniquely determined by measuring $y(t) = Cx(t)$ over a given time window $t \in [0, T]$.

Theorem (Popov-Belevitch-Hautus (PBH) test)

The linear system defined through (A, B) is controllable if and only if

$$\operatorname{rank}([A - \lambda I \quad B]) = n \qquad \forall \lambda \in \mathbb{C}$$

Theorem (PBH test)

The linear system (or equivalently the pair (A, C)) is observable if and only if

$$\operatorname{rank}\left(\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}\right) = n \qquad \forall \lambda \in \mathbb{C}$$

Note that:

- The rank of a matrix needs to be considered with caution. Example

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- Controllability/Observability is independent of the time interval $[0, T]$. In particular, T can be chosen arbitrarily small.

Systems with Inputs (Stabilizability)

Definition (Stabilizability)

Consider the linear system defined through the pair (A, B). The linear system (or equivalently the pair (A, B)) is said to be stabilizable, if for all $x \in \mathbb{R}^n$ there exists $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ such that

$$|x(t; x, u)| \to 0 \qquad \text{for} \qquad t \to \infty.$$

Intermediate step:

- Coordinate transformation, $T \in \mathbb{R}^{n \times n}$ invertible

$$\dot{x} = Ax + Bu, \qquad y = Cx + Du.$$
$$T\dot{x} = TAT^{-1}Tx + TBu, \quad y = CT^{-1}Tx + Du.$$

- With notation:

$\tilde{x} = Tx$, $\tilde{A} = TAT^{-1}$, $\tilde{B} = TB$, $\tilde{C} = CT^{-1}$

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u, \qquad y = \tilde{C}x + Du$$

- The coordinate transformation does not change the convergence properties i.e., $|x(t)| \to 0$ for $t \to \infty$ if and only if $|\tilde{x}(t)| \to 0$ for $t \to \infty$.

Proposition

Consider the pair (A, B). There exists an invertible matrix $T \in \mathbb{R}^{n \times n}$ such that

$$TAT^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \quad \text{and} \quad TB = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \qquad (2)$$

and the pair (A_{11}, B_1) is controllable.

Theorem

Consider the pair (A, B) together with the coordinate transformation (2) *where (A_{11}, B_1) is controllable. Then the pair (A, B) is stabilizable if and only if A_{22} is Hurwitz.*

Theorem (Modified PBH test)

The linear system (A, B) is stabilizable if and only if

$$\text{rank}\,([A - \lambda I \quad B]) = n, \qquad \lambda \in \overline{\mathbb{C}}_+ \qquad (3)$$

Lyapunov result: (A, B) is stabilizable $\Leftrightarrow$ $\exists P \in \mathcal{S}^n_{>0}$ so that $AP + PA^T - BB^T < 0$.

Systems with Inputs (Stabilizability)

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Systems with Inputs (Detectability)

Observability of (A, C) implies that for each $x_0 \neq 0$ there exists a $t \geq 0$ such that

$$Cx(t; x_0, 0) \neq Cx(t; 0, 0) = 0,$$

i.e., x_0 can be distinguished from 0.
If (A, C) is not observable define: (unobservable states)

$$\mathcal{N} = \{x_0 \in \mathbb{R}^n : Cx(t; x_0, 0) = 0 \ \forall t \geq 0\} \qquad (4)$$

Definition (Detectability)

Consider the linear system defined through the pair (A, C). The linear system with output (or equivalently the pair (A, C)) is said to be detectable, if for all $x_0 \in \mathcal{N}$ the solution satisfies

$$|x(t; x_0, 0)| \to 0 \quad \text{for} \quad t \to \infty.$$

Proposition

Consider the pair (A, C). There exists an invertible matrix $T \in \mathbb{R}^{n \times n}$ such that

$$TAT^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \qquad CT^{-1} = \begin{bmatrix} 0 & C_2 \end{bmatrix} \qquad (5)$$

and the pair (A_{22}, C_2) is observable.

Theorem

Consider the pair (A, C) together with the coordinate transformation (5) *where (A_{22}, C_2) is observable. Then the pair (A, C) is detectable if and only if A_{11} is Hurwitz.*

Theorem

The pair (A, C) is detectable if and only if

$$\text{rank}\left(\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}\right) = n, \qquad \lambda \in \overline{\mathbb{C}}_+$$

Lyapunov result: (A, C) is detectable $\Leftrightarrow$ $\exists P \in \mathcal{S}^n_{>0}$ so that $A^T P + PA - C^T C < 0$.

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Lyapunov result: (A, C) is detectable $\Leftrightarrow$ $\exists P \in \mathcal{S}^n_{>0}$ so that $A^T P + PA - C^T C < 0$.

Systems with Inputs (Kalman decomposition)

Proposition (Kalman decomposition)

Consider the linear system defined through (A, B, C, D)*. There exists an invertible matrix* $T \in \mathbb{R}^{n\times n}$ *such that*

$$TAT^{-1} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A_{22} & 0 & A_{24} \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{bmatrix}, \qquad TB = \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix},$$

$$CT^{-1} = \begin{bmatrix} 0 & C_2 & 0 & C_4 \end{bmatrix}$$

and such that

$$\left(\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \right)$$

is controllable and

$$\left(\begin{bmatrix} A_{22} & A_{24} \\ 0 & A_{44} \end{bmatrix}, \begin{bmatrix} C_2 & C_4 \end{bmatrix} \right)$$

is observable.

Systems with Inputs (Pole Placement, 1)

Consider

$$\dot{x} = Ax + Bu$$

For $u = 0$, asymptotic stability of $x^e = 0$ depends solely on the eigenvalues of A.
If A is not Hurwitz can we define $u = Kx$

$$\dot{x} = Ax + Bu = (A + BK)x$$

such that $A + BK$ is Hurwitz?

Theorem (Pole Placement)

Consider the linear system $\dot{x} = Ax + Bu$. Let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ with $\{\lambda_1, \ldots, \lambda_n\} = \{\overline{\lambda}_1, \ldots, \overline{\lambda}_n\}$. If (A, B) is controllable, then there exists a matrix $K \in \mathbb{R}^{m \times n}$ such that $\{\lambda_1, \ldots, \lambda_n\}$ is the set of eigenvalues of the closed loop matrix $A + BK$.

In Matlab:

- `acker.m`
- `place.m`

Example (Pendulum on a cart)

Linearization in the upright position:

$$A = \begin{bmatrix} 0 & 0 & 1.00 & 0 \\ 0 & 0 & 0 & 1.00 \\ 0 & 3.27 & -0.07 & -0.03 \\ 0 & 6.54 & -0.03 & -0.07 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ -0.67 \\ 0.33 \end{bmatrix}.$$

The eigenvalues of A (obtained using `eig.m` in Matlab):

$$\{0, 2.5162, -2.5995, -0.05\},$$

i.e., A is not Hurwitz. (Verify that (A, B) is controllable.)
With

$$K = \begin{bmatrix} 7.34 & -140.84 & 15.47 & -60.53 \end{bmatrix}$$

the closed loop matrix

$$A_{cl} = A + BK = \begin{bmatrix} 0 & 0 & 1.00 & 0 \\ 0 & 0 & 0 & 1.00 \\ 4.89 & -90.62 & 10.24 & -40.39 \\ 2.45 & -40.41 & 5.12 & -20.24 \end{bmatrix}$$

has eigenvalues $\{-1, -2, -3, -4\}$; i.e., $A_{cl}x$ is Hurwitz.

Systems with Inputs (Pole Placement, 2)

Consider

$$\dot{x} = Ax + Bu$$

For $u = 0$, asymptotic stability of $x^e = 0$ depends solely on the eigenvalues of A.
If A is not Hurwitz can we define $u = Kx$

$$\dot{x} = Ax + Bu = (A + BK)x$$

such that $A + BK$ is Hurwitz?

Theorem (Pole Placement)

Consider the linear system $\dot{x} = Ax + Bu$. Let $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ with $\{\lambda_1, \dots, \lambda_n\} = \{\overline{\lambda}_1, \dots, \overline{\lambda}_n\}$. If (A, B) is controllable, then there exists a matrix $K \in \mathbb{R}^{m \times n}$ such that $\{\lambda_1, \dots, \lambda_n\}$ is the set of eigenvalues of the closed loop matrix $A + BK$.

In Matlab:

- `acker.m`
- `place.m`

Pole placement for static output feedback:

$$\dot{x} = Ax + Bu, \qquad u = Ky$$
$$y = Cx$$

Closed loop system:

$$\dot{x} = (A + BKC)x$$

Theorem

If $\operatorname{trace}(A) > 0$ and $CB = 0$ then there is no matrix gain $K \in \mathbb{R}^{m \times p}$ such that $A + BKC$ is Hurwitz.

It holds that:

- $\operatorname{trace}(A) =$ sum of the eigenvalues of A
- $\operatorname{trace}(BKC) = \operatorname{trace}(CBK) = 0$
- $\operatorname{trace}(A + BKC) = \operatorname{trace}(A) + \operatorname{trace}(BKC)$ i.e., $\operatorname{trace}(A + BKC) = \operatorname{trace}(A) > 0$
- $A + BKC$ has at least one eigenvalue in the right half plane.

Introduction to Nonlinear Control

Stability, control design, and estimation

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Part I:

Chapter 3: Linear Systems and Linearization

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