Introduction to Nonlinear Control

Stability, control design, and estimation

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Part I:

Chapter 4: Frequency Domain Analysis

4.1 Fundamental Results in the Frequency Domain
4.2 The Transfer Function

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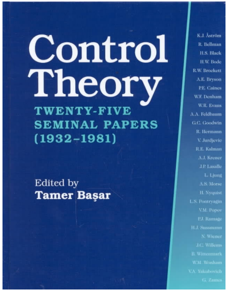
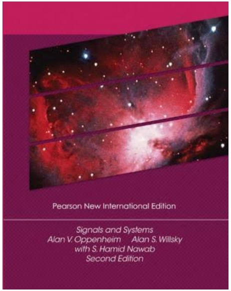
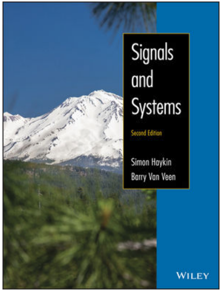
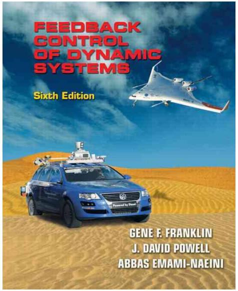
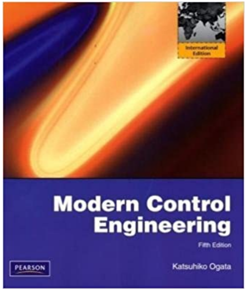
Frequency Domain Analysis











Frequency Domain Analysis

1. Fundamental Results in the Frequency Domain
   - The Laplace Transform
   - The Transfer Functions
   - The $\mathcal{L}_2$-, $\mathcal{L}_\infty$- and $\mathcal{H}_\infty$-norm
2. Stability Analysis in the Frequency Domain
   - Bounded-Input, Bounded-Output Stability
   - System Interconnections in the Frequency Domain
   - The Bode Plot
   - The Nyquist Criterion

Section 1

Fundamental Results in the Frequency Domain

Fundamental Results in the Frequency Domain, 1

Consider single-input single-output (SISO) linear systems:

$$\dot{x}(t) = Ax(t) + bu(t), \qquad y(t) = cx(t) + du(t),$$

Frequency domain representation:

$$\hat{y}(s) = G(s)\hat{u}(s)$$

Notation and assumptions:

- Transfer function $G : \mathbb{C} \to \mathbb{C}$.
- G is a rational function, i.e., there exist polynomial functions $P, Q \in \mathbb{R}[s]$ (with coefficients in $\mathbb{R}$) such that

$$G(s) = \frac{P(s)}{Q(s)}.$$

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- Consider $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$, $m \in \mathbb{N}$, and the integral $\int_0^\infty \psi(t)e^{-st}\, dt$ for $s \in \mathbb{C}$ for which the integral is well-defined

Definition (Laplace transform)

Consider $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$. For $s \in \mathcal{C} \subset \mathbb{C}$ for which the integral is well-defined, the Laplace transform $\hat{\psi} : \mathcal{C} \to \mathbb{C}^m$ of ψ is defined as

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Example: Consider $\psi(t) = 1$. For fixed $s \in \mathbb{C}$ compute

$$\int_0^\infty e^{-st}\, dt = -\frac{1}{s}e^{-st}\Big|_0^\infty = \frac{1}{s}$$

i.e., $\hat{\psi}(s) = (\mathscr{L}\psi)(s) = \frac{1}{s}$.

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Definition (Inverse Laplace transform)

Consider $\hat{\varphi} : \mathcal{C} \to \mathbb{C}^m$ and let $\alpha \in \mathbb{R}$ such that $\alpha + j\beta \in \mathcal{C} \subset \mathbb{C}$ for all $\beta \in \mathbb{R}$. Then the inverse Laplace transform $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ of $\hat{\varphi}$ is defined as

$$\varphi(t) \doteq (\mathscr{L}^{-1}\hat{\varphi})(t) \doteq \frac{1}{2\pi j}\int_{\alpha - j\infty}^{\alpha + j\infty} e^{st}\hat{\varphi}(s)\, ds$$

$$= \frac{e^{\alpha t}}{2\pi j}\int_{-\infty}^{\infty} e^{jwt}\hat{\varphi}(\alpha + jw)\, dw.$$

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Proposition (Laplace transform properties)

Consider the signals $\varphi, \varphi_1, \varphi_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ in the time domain and constants $a \in \mathbb{R}_{>0}$, $a_1, a_2 \in \mathbb{R}$. Then the Laplace transform and its inverse satisfy the following properties:

$$\mathscr{L}^{-1}\mathscr{L}\varphi(t) = \varphi(t),$$
$$\mathscr{L}(a_1\varphi + a_2\varphi_2)(s) = a_1\hat{\varphi}_1(s) + a_2\hat{\varphi}_2(s),$$
$$\mathscr{L}(\varphi(a\cdot))(s) = \tfrac{1}{a}\hat{\varphi}\left(\tfrac{s}{a}\right),$$
$$\mathscr{L}(\varphi(\cdot - a))(s) = e^{-sa}\hat{\varphi}(s),$$
$$\mathscr{L}(\tfrac{d^k}{dt^k}\varphi)(s) = s^k\hat{\varphi}(s) - \textstyle\sum_{j=1}^{k-1} s^{j-1}\tfrac{d^{k-1-j}}{dt^{k-1-j}}\varphi(0),$$
$$\mathscr{L}\left(\textstyle\int_0^{\cdot}\varphi(\tau)\,d\tau\right)(s) = \tfrac{1}{s}\hat{\varphi}(s).$$

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$$\hat{y}(s) = \left(c(sI - A)^{-1}b + d\right)\hat{u}(s)$$

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$$G(s) = \frac{\hat{y}(s)}{\hat{u}(s)} = c(sI - A)^{-1}b + d \qquad (1)$$

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Consider a transfer function $G(s)$ and assume that (1) is satisfied for (A, b, c, d). Then $G(s)$ is called realizable and the quadruple (A, b, c, d) is called a realization of $G(s)$.

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Consider a transfer function $G(s) = \frac{P(s)}{Q(s)}$, $P, Q \in \mathbb{R}[s]$.
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Theorem (Uncontrollable & unobs. modes)

Let (A, b, c, d) be a realization of $G(s) = \frac{P(s)}{Q(s)}$. If $\lambda \in \mathbb{C}$ is a pole of G, i.e., $Q(\lambda) = 0$, then λ is an eigenvalue of A. Conversely, let λ be an eigenvalue of A such that $G(\lambda) \neq 0$, then λ is an uncontrollable mode of (A, b) or an unobservable mode of (A, c).

The $\mathcal{L}_2$-, $\mathcal{L}_\infty$- and $\mathcal{H}_\infty$-norm

Consider $\psi : [0,t) \to \mathbb{R}^n$, $n \in \mathbb{N}$, for $t \in \mathbb{R}_{\geq 0} \cup \{\infty\}$.

- $\mathcal{L}_2$-norm: For ψ with $(\int_0^t |\psi(\tau)|^2 \, d\tau)^{\frac{1}{2}} < \infty$, define the norm

$$\|\psi\|_{\mathcal{L}_2[0,t)} \doteq \left(\int_0^t |\psi(\tau)|^2 \, d\tau \right)^{\frac{1}{2}}$$

- $\mathcal{L}_\infty$-norm: For essentially bounded functions ψ, define the norm

$$\|\psi\|_{\mathcal{L}_\infty[0,t)} \doteq \operatorname*{ess\,sup}_{\tau \in [0,t)} |\psi(\tau)|$$

$$\doteq \inf\{\eta \in \mathbb{R}_{\geq 0} : |\psi(t)| \leq \eta \text{ for almost all } \tau \in [0,t)\}$$

Note that:

- Two norms are combined in the definitions: $\|\cdot\|_{\mathcal{L}_2}$, $\|\cdot\|_{\mathcal{L}_\infty}$ define norms of a function $\psi(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ and $|\cdot|$ denotes a vector norm $\psi(t) \in \mathbb{R}^n$ for a fixed $t \in \mathbb{R}_{\geq 0}$. For $x \in \mathbb{C}$, $|x| = \sqrt{\overline{x}^T x}$.

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Proposition (Parseval's theorem)

Consider a signal $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ in the time domain satisfying $\|\psi\|_{\mathcal{L}_2[0,\infty)} < \infty$ and its Laplace transform $\hat{\psi} : \mathbb{C} \to \mathbb{C}^n$. Then Parseval's relation

$$\int_0^\infty |\psi(\tau)|^2 \, d\tau = \frac{1}{2\pi} \int_{-\infty}^\infty |\hat{\psi}(j\omega)|^2 \, d\omega$$

is satisfied.

(Relation between $\mathcal{L}_2$-norm and Laplace transform)

Section 2

Stability Analysis in the Frequency Domain

Stability Analysis in the Frequency Domain (Bounded-Input, Bounded-Output Stability)

Consider

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Definition (BIBO stability)

The linear system is called bounded-input, bounded-output (BIBO) stable if $\|u\|_{\mathcal{L}_\infty} < \infty$ implies $\|y\|_{\mathcal{L}_\infty} < \infty$.

It holds that:

- The linear system is BIBO stable if and only if there exists $\eta \in \mathbb{R}_{>0}$ such that

$$\|y\|_{\mathcal{L}_\infty} \leq \eta \|u\|_{\mathcal{L}_\infty}, \qquad \forall\, u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$$

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Assume that the origin of the linear system with zero-input is exponentially/asymptotically stable. Then the system is BIBO stable.

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Note that:

- The converse is not true. Example: Let $c = 0$ (and $d = 0$). Then $y(t) \doteq 0$ for all inputs $u(\cdot)$ i.e., the system is BIBO stable (independent of A and b).

Lemma

Consider the transfer function $G(s)$ and an arbitrary realization (A, b, c, d). Then the system in the frequency domain and the corresponding system in the time domain are BIBO stable if and only if all poles of $G(s)$ are in $\mathbb{C}_-$.

System Interconnections in the Frequency Domain

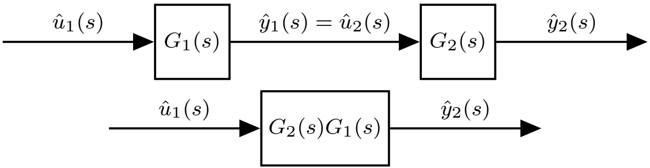
Consider two systems:

$$\hat{y}_1(s) = G(s)\hat{u}_1(s)$$
$$\hat{y}_2(s) = G(s)\hat{u}_2(s)$$

Cascade interconnection

$$\hat{y}_2(s) = G_2(s)G_1(s)\hat{u}_1(s)$$

Cascade interconnection $\hat{u}_2(s) = \hat{y}_1(s)$

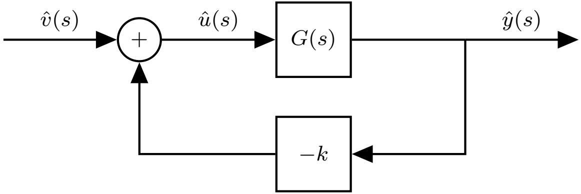


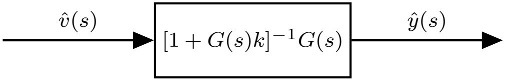
System Interconnections in the Frequency Domain

Consider:

$$\hat{y}(s) = G(s)\hat{u}(s)$$
$$\hat{u}(s) = \hat{v}(s) - k\hat{y}(s)$$

Feedback interconnection:





System Interconnections in the Frequency Domain

Consider:

$$\hat{y}(s) = G(s)\hat{u}(s)$$
$$\hat{u}(s) = \hat{v}(s) - k\hat{y}(s)$$

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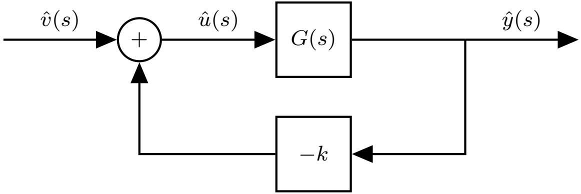

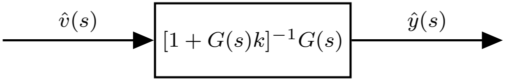

Rewrite the input: (new input $v : \mathbb{R}_{\geq 0} \to \mathbb{R}$)

$$u(t) = v(t) - ky(t), k \in \mathbb{R}$$

The Laplace transform

$$\hat{u}(s) = \hat{v}(s) - k\hat{y}(s).$$

Thus

$$\hat{y}(s) = G(s)(\hat{v}(s) - k\hat{y}(s))$$
$$\hat{y}(s) = \frac{G(s)}{1 + G(s)k}\hat{v}(s)$$

BIBO stability can be guaranteed by selecting the feedback gain k such that the closed loop transfer function only has poles in the open left halfplane $\mathbb{C}_-$.

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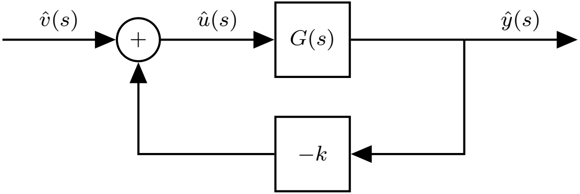

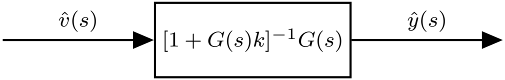

Example

Consider:

$$G(s) = \frac{1}{s^2 + 0.1s - 9.81}$$

Poles: $\lambda_1 = -3.1825$ and $\lambda_2 = 3.0825$.
Feedback interconnection for $k \in \mathbb{R}$:

$$[1+G(s)k]^{-1}G(s) = \frac{\frac{1}{s^2+0.1s-9.81}}{1+\frac{k}{s^2+0.1s-9.81}} = \frac{1}{s^2 + 0.1s - 9.81 + k}.$$

Poles for $k = 10$: $-0.05 \pm 0.433j \in \mathbb{C}_-$

The Bode Plot

Consider a BIBO stable system:

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$$u(t) = \sin(\omega t), \qquad t \geq 0$$

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  - (Recall that $\mathcal{H}_\infty$-norm captures the maximal amplification of a signal)
- The Bode Plot visualizes $|G(j\omega)|$ and $\varphi(\omega)$ over $\omega \in \mathbb{R}$ on a $\log_{10}$/$\log_{10}$-scale and a $\log_{10}$/linear-scale

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Example: Linearization of the inverted pendulum around the stable equilibrium $[x_1, x_2]^T = [\theta, \dot{\theta}]^T = [\pi, 0]^T$.

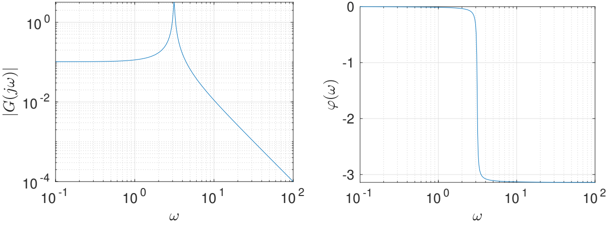
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{mg\ell}{J+m\ell^2} & -\frac{\gamma}{J+m\ell^2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\ell}{J+m\ell^2} \end{bmatrix} u,$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x.$$

Transfer function with $m = \ell = 1$, $J = 0$, $g = 9.81$, $\gamma = 0.1$:

$$G(s) = \frac{P(s)}{Q(s)} = \frac{1}{s^2 + 0.1s + 9.81}.$$

The Bode Plot:

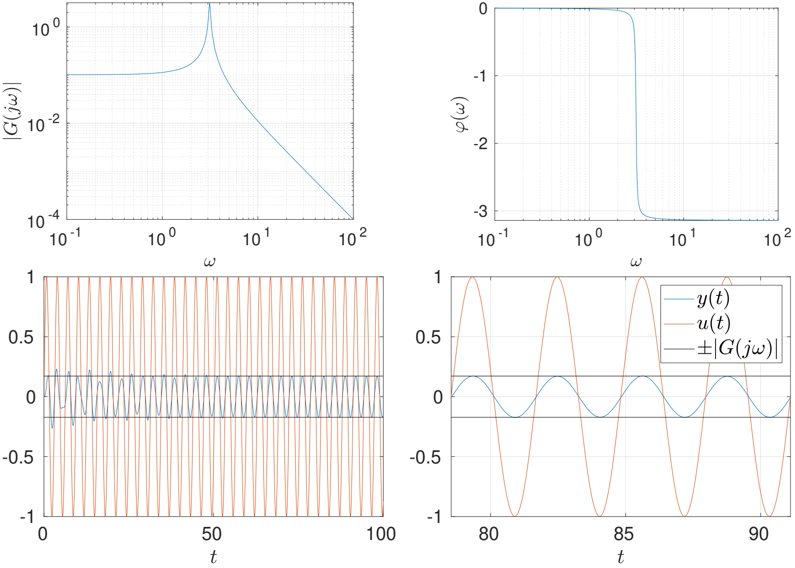


The Bode Plot (Example Continued)

Input-output behavior for

$$u(t) = \sin(\omega t)$$

- $\omega = 2$



The Bode Plot (Example Continued)

Input-output behavior for

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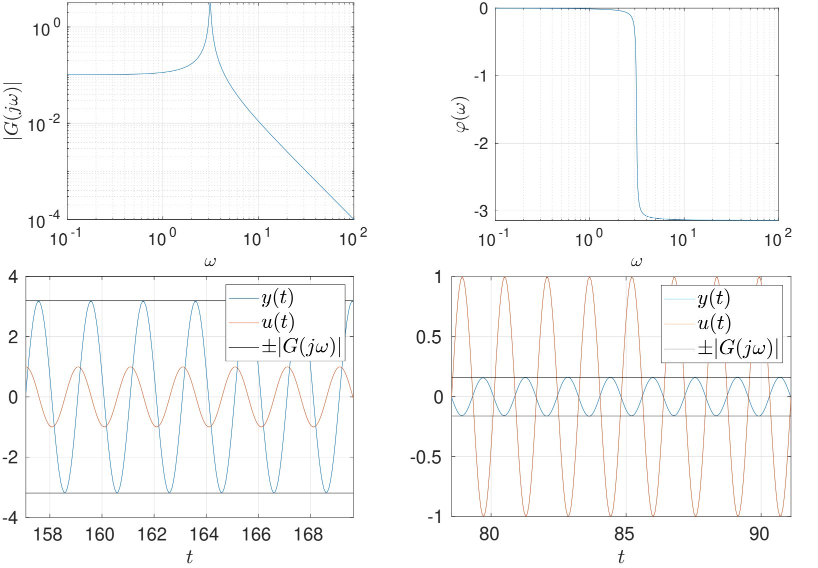
- $\omega = 3.13$
- $\omega = 4$



The Bode Plot (Example Continued)

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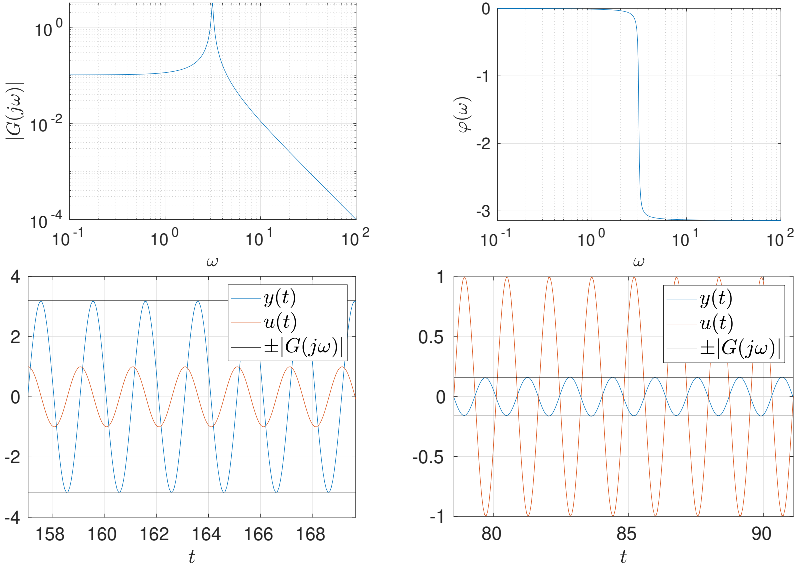
Note that

- the Bode plot is used to experimentally derive the transfer function
- the magnitude is usually shown in dB (decibel):

$$|G(jw)| \iff \log_{10} |G(jw)| dB$$

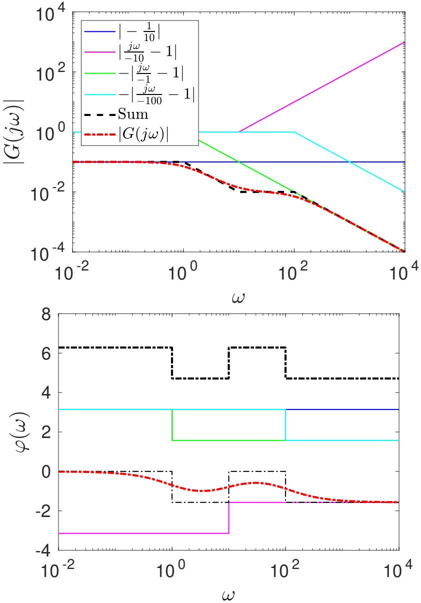


The Bode Plot (Example: Sketching the Bode Plot)

Example:

Consider

$$G(s) = \frac{P(s)}{Q(s)} = \frac{s+10}{s^2+101s+100} \quad \rightsquigarrow \quad G(s) = c\frac{\prod_{i=1}^{d_P}(\frac{s}{p_i}-1)}{\prod_{j=1}^{d_Q}(\frac{s}{q_j}-1)}$$



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In terms of the logarithm:

$$\log_{10}|G(j\omega)| = \log_{10}(|c|) + \sum_{i=1}^{d_P}\log_{10}\left(\left|\frac{j\omega}{p_i}-1\right|\right) - \sum_{i=1}^{d_Q}\log_{10}\left(\left|\frac{j\omega}{q_i}-1\right|\right)$$

Approximation of the individual terms:

$$\omega \text{ small} \quad \Rightarrow \quad \log_{10}\left(\left|\frac{j\omega}{\kappa}-1\right|\right) \approx \log_{10}(1) = 0$$

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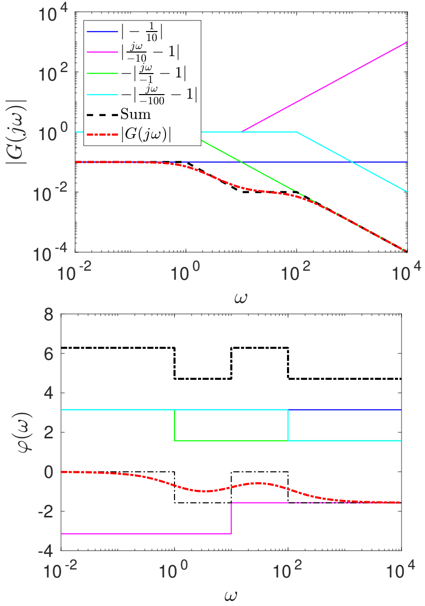

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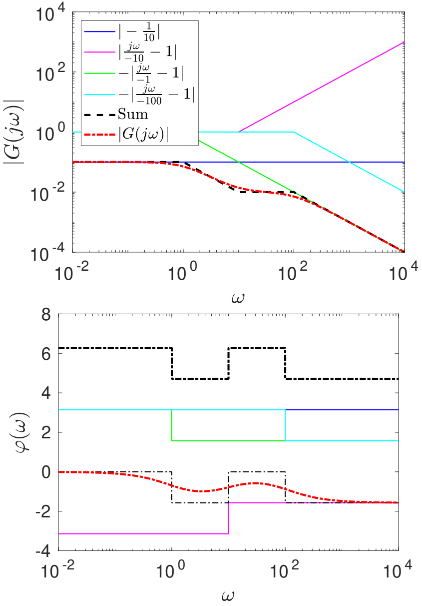
Similar, decomposition of the phase:

$$\varphi = \arctan_2(0,c) + \sum_{i=1}^{d_P}\arctan_2\left(\frac{\omega}{p_i},-1\right) - \sum_{i=1}^{d_Q}\arctan_2\left(\frac{\omega}{q_i},-1\right)$$



The Nyquist Criterion

Consider: SISO feedback interconnection

$$\hat{y}(s) = G_{cl}(s)\hat{u}(s) = \frac{G_{ol}(s)}{1 + kG_{ol}(s)}\hat{u}(s)$$

where

- G_{ol} open loop transfer function
- G_{cl} closed loop transfer function

Recall that:

- The system is BIBO stable $\Longleftrightarrow$ G_{cl} does not have any poles in $\overline{\mathbb{C}}_+$
- The zeros of $1 + kG_{ol}(s)$ are the poles of $G_{cl}(s)$

Thus, for BIBO stability of G_{cl} we require

- $1 + kG_{ol}(j\omega) \neq 0$ or $G_{ol}(j\omega) \neq -1/k$
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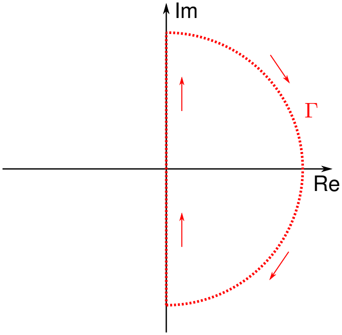
Cauchy's Argument Principle:

$$w_n = -\frac{1}{2\pi j}\oint_\Gamma \frac{g'(s)}{g(s)} ds = Z - P,$$

• w_n winding number; • Z and P: zeros/poles of $g(\cdot)$ contained within Γ.

Now, BIBO stability requires $Z = 0$ so that $P = \frac{1}{2\pi j}\oint_\Gamma \frac{kG'_{ol}(s)}{1+kG_{ol}(s)} ds$.

The Nyquist plot is a graphical representation of the transfer function evaluated along a closed contour Γ in $\mathbb{C}$ that traverses the imaginary axis and a semicircle of infinite radius.



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Theorem (Nyquist Criterion)

Consider the SISO closed loop system. Let $P \in \mathbb{N}$ denote the number of poles of G_{ol} in $\mathbb{C}^+$. Moreover, assume that G_{ol} does not have any poles in $j\mathbb{R}$. Then the system is BIBO stable if and only if $G_{ol}(jw)$, $w \in [-\infty, \infty]$, encircles $-1/k \in \mathbb{C}$ exactly $-P$-times clockwise.

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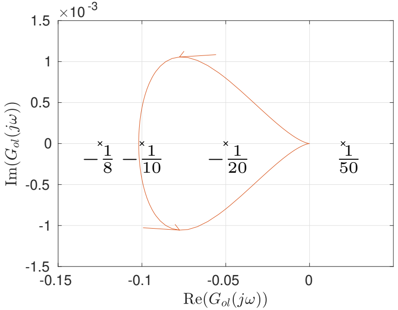

For $k < 9.81$ the graph of $G_{ol}(j\omega)$ encircles the point $-\frac{1}{k}$ zero times and for $k > 9.81$ the graph encircles the point $-\frac{1}{k}$ exactly -1 time clockwise.

Introduction to Nonlinear Control

Stability, control design, and estimation

Philipp Braun & Christopher M. Kellett
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Australian National University, Canberra, Australia

Part I:

Chapter 4: Frequency Domain Analysis

Australian National University