Introduction to Nonlinear Control

Stability, control design, and estimation

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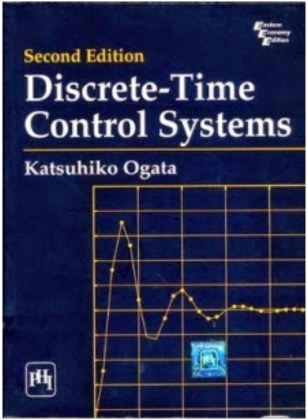
Part I:

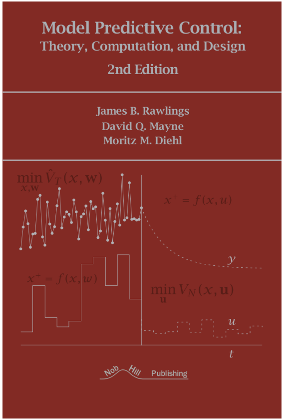
Chapter 5: Discrete Time Systems

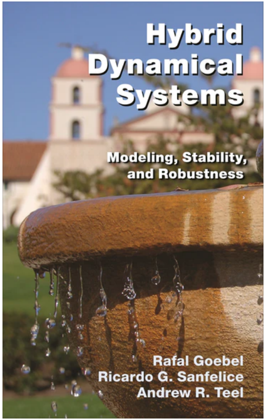
5.1 Discrete Time Systems – Fundamentals
5.2 Sampling From Continuous to Discrete Time
5.3 Stability Notions
5.4 Controllability and Observability

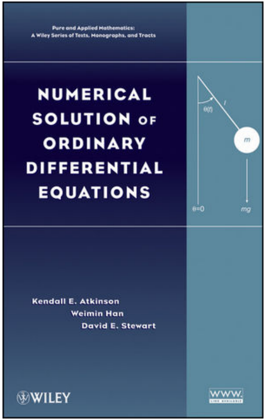
Australian National University

Discrete Time Systems









Nonlinear Systems - Fundamentals

1. Discrete Time Systems – Fundamentals
2. Sampling: From Continuous Time to Discrete Time
   - Discretization of Linear Systems
   - Higher Order Discretization Schemes
3. Stability Notions
   - Lyapunov Characterizations
   - Linear Systems
   - Stability Preservation of Discretized Systems
4. Controllability and Observability

Section 1

Discrete Time Systems – Fundamentals

Discrete Time Systems – Fundamentals

Discrete time sys. $(F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n,\ H : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p)$

$$
\begin{aligned}
x_d(k+1) &= F(x_d(k), u_d(k)), \quad x_d(0) = x_{d,0} \in \mathbb{R}^n \\
y_d(k) &= H(x_d(k), u_d(k))
\end{aligned}
$$

Discrete Time Systems – Fundamentals

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Time-varying discrete time system ($k \geq k_0 \geq 0$):

$$x_d(k+1) = F(k, x_d(k)), \quad x_d(k_0) = x_{d,0} \in \mathbb{R}^n$$

Time invariant discrete time systems without input:

$$x_d(k+1) = F(x_d(k)), \quad x_d(0) = x_{d,0} \in \mathbb{R}^n,$$

Shorthand notation for difference equations:

$$x_d^+ = F(x_d, u_d),$$

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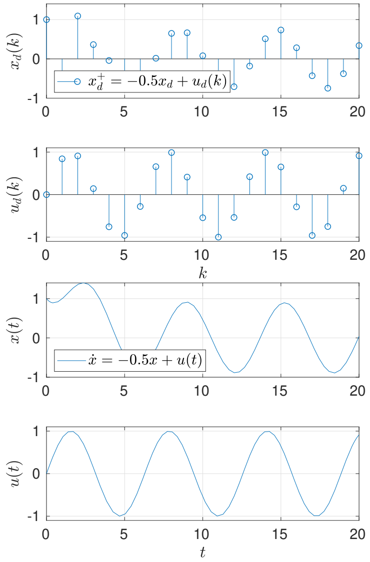

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Definition (Equilibrium)

- The point $x_d^e \in \mathbb{R}^n$ is called equilibrium if $x_d^e = F(x_d^e)$ or $x_d^e = F(k, x_d^e)$ for all $k \in \mathbb{N}$ is satisfied.
- The pair $(x_d^e, u_d^e) \in \mathbb{R}^n \times \mathbb{R}^m$ is called equilibrium pair of the system if $x_d^e = F(x_d^e, u_d^e)$ holds.

Again, without loss of generality we can shift the equilibrium (pair) to the origin.

Definition (Equilibrium, $\dot{x} = 0$)

The point $x^e \in \mathbb{R}^n$ is called an equilibrium of the system $\dot{x} = f(x)$ if $\frac{d}{dt}x(t) = f(x^e) = 0$

Section 2

Sampling: From Continuous Time to Discrete Time

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Derivative for continuously differentiable function:

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$$\frac{x(t+\Delta) - x(t)}{\Delta} \approx \frac{d}{dt}x(t) = \dot{x}(t) = f(x(t), u(t))$$

or equivalently

$$x(t+\Delta) \approx x(t) + \Delta f(x(t), u(t))$$

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Approximated discrete time system (identify t with $k \cdot \Delta$)

$$x_d^+ = F(x_d, u_d) \doteq x_d + \Delta f(x_d, u_d)$$

$\rightsquigarrow$ This discretization is known as (explicit) *Euler method*.

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- Continuous time: $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ and $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$
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Zero-order hold: for all $k \in \mathbb{N}$, for all $t \in [0, \Delta)$

$$x_d(k) = x(\Delta k) = x(t + \Delta k)$$
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(restrict x and u to piecewise constant functions)

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Sample-and-hold input: (with sampling rate Δ)

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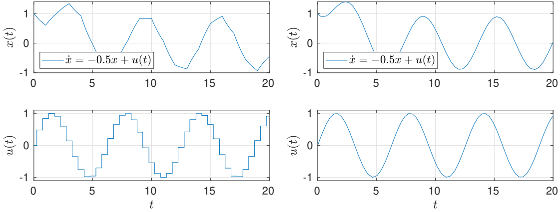
Digital controller:

- apply a piecewise constant sample-and-hold input to a continuous time system.

Solution corresponding to sample-and-hold input ($\Delta = 1$) and continuous input



Discretization of Linear Systems

Consider the linear system: ($A_c \in \mathbb{R}^{n\times n}$, $B_c \in \mathbb{R}^{n\times m}$)

$$\dot{x}(t) = A_c x(t) + B_c u(t)$$

Euler discretization: (sampling rate $\Delta > 0$)

$$\begin{aligned} x(t+\Delta) &\approx x(t) + \Delta(A_c x(t) + B_c u(t)) \\ &= (I + \Delta A_c)x(t) + \Delta B_c u(t) \end{aligned}$$

Linear discrete time system:

$$x_d(k+1) = A_d x_d(k) + B_d u_d(k)$$
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Alternative discretization (for linear systems):
Recall the solution of the linear system:

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Then

$$x(t+\Delta) = e^{A_c\Delta}x(t) + \int_0^\Delta e^{A_c(\Delta-\tau)}d\tau B_c u(t).$$

Define

$$A_{de} \doteq e^{A_c\Delta} \qquad \text{and} \qquad B_{de} \doteq \int_0^\Delta e^{A_c(\Delta-\tau)}d\tau B_c$$

Alternative discrete time system:

$$x_d^+ = A_{de}x_d + B_{de}u_d$$

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The discretization satisfies

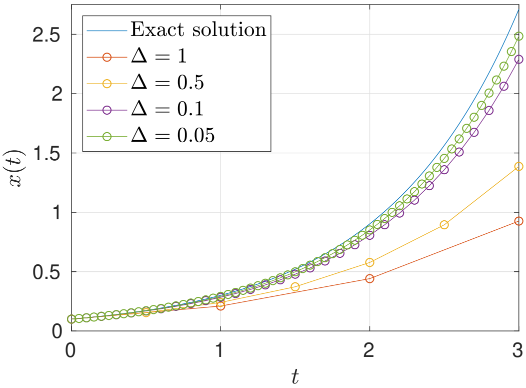
$$x(k\Delta) = x_d(k), \qquad \text{for all } k \in \mathbb{N}$$

if $u(t + \Delta k) = u(\Delta k) = u_d(k)\ \forall\, t \in [0, \Delta),\ \forall\, k \in \mathbb{N}$.

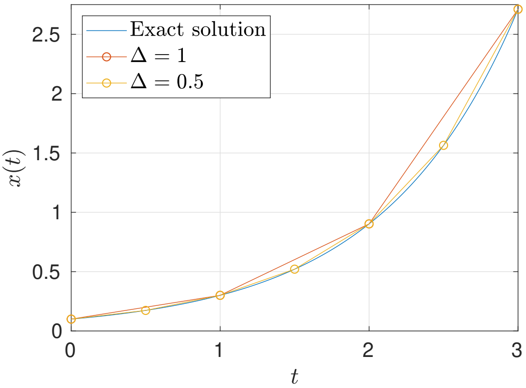
Discretization of Linear Systems (Comparison)

Approximation of $\dot{x} = 1.1x$

Euler discretization: $x^+ = (I + \Delta A_c)x$



Exact discretization: $x^+ = e^{A_c \Delta}x$



Higher Order Discretization Schemes

Consider the continuous time system

$$\dot{x}(t) = f(x(t), u(t))$$

Assume that f is sufficiently often cont. differentiable:

$$\frac{d^{i+1}}{dt^{i+1}} x(t) = \frac{d^i}{dt^i} f(x(t), u(t)), \qquad i = 1, \ldots, r$$

Taylor approximation of $x(t)$:

$$x(t+\Delta) = x(t) + \dot{x}(t)\Delta + \tfrac{1}{2!}\ddot{x}(t)\Delta^2 + \cdots + \tfrac{1}{r!}\tfrac{d^r}{dt^r} x(t)\Delta^r + R_r(\Delta)$$

Remainder

$$R_r(\Delta) = \tfrac{1}{(r+1)!}\tfrac{d^{r+1}}{dt^{r+1}} x(\tau)\Delta^{r+1}, \qquad \tau \in [t, t+\Delta]$$

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Example: Consider $r = 1$. Then

$$\begin{aligned} x(t+\Delta) &= x(t) + \dot{x}(t)\Delta + R_1(\Delta) \\ &= x(t) + \Delta f(x(t)) + R_1(\Delta) \end{aligned}$$

Moreover, $R_1(\Delta) \overset{\Delta \to 0}{\longrightarrow} 0$, quadratically.

$\rightsquigarrow$ The Euler method converges quadratically.

Include higher order terms in the approximation:

$$x(t+\Delta) \approx x(t) + \dot{x}(t)\Delta + \tfrac{1}{2}\ddot{x}(t)\Delta^2$$

(We ignore terms of order Δ^3 in the remainder.)
Note that

$$\begin{aligned} \ddot{x} = \tfrac{df}{dt}(x,u) &= \tfrac{\partial}{\partial x} f(x,u)\dot{x} + \tfrac{\partial}{\partial u} f(x,u)\dot{u}, \\ &= \tfrac{\partial}{\partial x} f(x,u) f(x,u) + \tfrac{\partial}{\partial u} f(x,u)\dot{u}, \end{aligned}$$

and thus

$$\begin{aligned} &x(t+\Delta) \approx x(t) + \Delta f(x(t), u(t)) \\ &\quad + \tfrac{\Delta^2}{2}\left(\tfrac{\partial}{\partial x} f(x(t), u(t)) f(x(t), u(t)) + \tfrac{\partial}{\partial u} f(x(t), u(t))\dot{u}(t)\right) \end{aligned}$$

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If $u(t)$ is piecewise constant we simplify to:

$$x(t+\Delta) \approx x(t) + \Delta f(x(t), u_d) + \tfrac{\Delta^2}{2}\tfrac{\partial}{\partial x}f(x(t), u_d)f(x(t), u_d)$$

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$$\dot{x}(t) = f(x(t), u(t))$$

Assume that f is sufficiently often cont. differentiable:

$$\frac{d^{i+1}}{dt^{i+1}} x(t) = \frac{d^i}{dt^i} f(x(t), u(t)), \qquad i = 1, \ldots, r$$

Taylor approximation of $x(t)$:

$$x(t+\Delta) = x(t) + \dot{x}(t)\Delta + \tfrac{1}{2!}\ddot{x}(t)\Delta^2 + \cdots + \tfrac{1}{r!}\tfrac{d^r}{dt^r} x(t)\Delta^r + R_r(\Delta)$$

Remainder

$$R_r(\Delta) = \tfrac{1}{(r+1)!}\tfrac{d^{r+1}}{dt^{r+1}} x(\tau)\Delta^{r+1}, \qquad \tau \in [t, t+\Delta]$$

Example: Consider $r = 1$. Then

$$\begin{aligned} x(t+\Delta) &= x(t) + \dot{x}(t)\Delta + R_1(\Delta) \\ &= x(t) + \Delta f(x(t)) + R_1(\Delta) \end{aligned}$$

Moreover, $R_1(\Delta) \overset{\Delta\to 0}{\longrightarrow} 0$, quadratically.

$\rightsquigarrow$ The Euler method converges quadratically.

Include higher order terms in the approximation:

$$x(t+\Delta) \approx x(t) + \dot{x}(t)\Delta + \tfrac{1}{2}\ddot{x}(t)\Delta^2$$

(We ignore terms of order Δ^3 in the remainder.)
Note that

$$\begin{aligned} \ddot{x} = \tfrac{df}{dt}(x,u) &= \tfrac{\partial}{\partial x} f(x,u)\dot{x} + \tfrac{\partial}{\partial u} f(x,u)\dot{u}, \\ &= \tfrac{\partial}{\partial x} f(x,u) f(x,u) + \tfrac{\partial}{\partial u} f(x,u)\dot{u}, \end{aligned}$$

and thus

$$x(t+\Delta) \approx x(t) + \Delta f(x(t), u(t)) + \tfrac{\Delta^2}{2}\left(\tfrac{\partial}{\partial x} f(x(t),u(t)) f(x(t),u(t)) + \tfrac{\partial}{\partial u} f(x(t),u(t))\dot{u}(t)\right)$$

If $u(t)$ is piecewise constant we simplify to:

$$x(t+\Delta) \approx x(t) + \Delta f(x(t), u_d) + \tfrac{\Delta^2}{2}\tfrac{\partial}{\partial x} f(x(t), u_d) f(x(t), u_d)$$

Avoid the calculation of $\frac{\partial f}{\partial x}$:

$$f(x + \Delta\dot{x}, u_d) = f(x, u_d) + \tfrac{\partial f}{\partial x}(x, u_d)\dot{x}\Delta + \tfrac{1}{2}\tfrac{d^2 f}{d\Delta^2}(x + \delta\dot{x}, u_d)\Delta^2 \quad \text{for a } \delta \in [0, \Delta]$$

Higher Order Discretization Schemes (2)

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Rearranging the terms:

$$\begin{aligned}\Delta\tfrac{\partial f}{\partial x}(x,u_d) f(x,u_d) &= f(x+\Delta f(x,u_d),u_d) - f(x,u_d) \\ &- \tfrac{1}{2}\tfrac{d^2 f}{d\Delta^2}(x+\delta\dot{x},u_d)\Delta^2.\end{aligned}$$

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$$\begin{aligned}x(t+\Delta) &\approx x(t) + \Delta f(x(t),u_d) \\ &\quad + \tfrac{\Delta}{2}\left(f(x(t)+\Delta f(x(t),u_d),u_d) - f(x(t),u_d)\right) \\ &\quad + \tfrac{\Delta}{2}\left(-\tfrac{1}{2}\tfrac{d^2 f}{d\Delta^2}(x(t)+\delta\dot{x},u_d)\Delta^2\right) \\ &= x(t) + \tfrac{\Delta}{2} f(x(t),u_d) + \tfrac{\Delta}{2} f(x(t)+\Delta f(x(t),u_d),u_d) \\ &\quad - \tfrac{1}{4}\tfrac{d^2 f}{d\Delta^2}(x(t)+\delta\dot{x},u_d)\Delta^3\end{aligned}$$

$\rightsquigarrow$ Ignore terms of order Δ^3

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Continuing with the approximation:

$$\begin{aligned}x(t+\Delta) &\approx x(t) + \Delta f(x(t),u_d) \\ &+ \tfrac{\Delta}{2}\left(f(x(t)+\Delta f(x(t),u_d),u_d) - f(x(t),u_d)\right) \\ &+ \tfrac{\Delta}{2}\left(-\tfrac{1}{2}\tfrac{d^2 f}{d\Delta^2}(x(t)+\delta\dot{x},u_d)\Delta^2\right) \\ &= x(t) + \tfrac{\Delta}{2} f(x(t),u_d) + \tfrac{\Delta}{2} f(x(t)+\Delta f(x(t),u_d),u_d) \\ &- \tfrac{1}{4}\tfrac{d^2 f}{d\Delta^2}(x(t)+\delta\dot{x},u_d)\Delta^3\end{aligned}$$

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Heun method:

$$\begin{aligned}x(t+\Delta) &\approx x(t) + \tfrac{\Delta}{2} f(x(t),u_d) \\ &+ \tfrac{\Delta}{2} f(x(t)+\Delta f(x(t),u_d),u_d)\end{aligned}$$

$\rightsquigarrow$ Cubic convergence

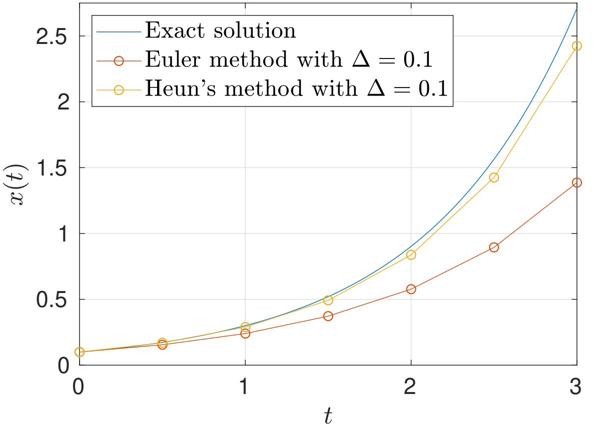
Comparison: Euler & Heun Method

- Consider $\dot{x} = 1.1x$
- Euler method:

$$x(t+\Delta) \approx x(t) + \Delta f(x(t), u_d)$$

- Heun method:

$$\begin{aligned} x(t+\Delta) \approx x(t) &+ \tfrac{\Delta}{2} f(x(t), u_d) \\ &+ \tfrac{\Delta}{2} f(x(t) + \Delta f(x(t), u_d), u_d) \end{aligned}$$



Runge-Kutta Methods

- Consider

$$\dot{x} = g(t, x).$$

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- Runge-Kutta update formula:

$$x(t+\Delta) = x(t) + \Delta \sum_{i=1}^{s} b_i k_i$$

where

$$\begin{aligned}
k_1 &= g(t, x(t)) \\
k_2 &= g(t + c_2\Delta, x + \Delta(a_{21}k_1)) \\
k_3 &= g(t + c_3\Delta, x + \Delta(a_{31}k_1 + a_{32}k_2)) \\
&\vdots \\
k_s &= g(t + c_s\Delta, x + \Delta(a_{s1}k_1 + a_{s2}k_2 + \cdots + a_{s(s-1)}k(s)))
\end{aligned}$$

- $s \in \mathbb{N}$ (stage); $a_{ij}, b_\ell, c_i \in \mathbb{R}$, $1 \leq j < i \leq s$, $1 \leq \ell \leq s$ (given parameters)

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- The case $f(x, u)$ for sample-and-hold inputs $u(t + \delta) = u_d \in \mathbb{R}^m$ for all $\delta \in [0, \Delta)$ is covered through

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- Butcher tableau:

$$\begin{array}{c|ccccc} 0 & & & & & \\ c_2 & a_{21} & & & & \\ c_3 & a_{31} & a_{32} & & & \\ \vdots & \vdots & & \ddots & & \\ c_s & a_{s1} & a_{s2} & \cdots & a_{s(s-1)} & \\ \hline & b_1 & b_2 & \cdots & b_{s-1} & b_s \end{array}$$

$\rightsquigarrow$ c_i is only necessary for time-varying systems

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\vdots & \vdots & & \ddots & & \\
c_s & a_{s1} & a_{s2} & \cdots & a_{s(s-1)} & \\
\hline
 & b_1 & b_2 & \cdots & b_{s-1} & b_s
\end{array}$$

$\rightsquigarrow$ c_i is only necessary for time-varying systems

- Examples: The Euler and the Heun method

$$\begin{array}{c|c}
0 & \\
\hline
 & 1
\end{array}
\quad \text{and} \quad
\begin{array}{c|cc}
0 & & \\
1 & 1 & \\
\hline
 & \frac{1}{2} & \frac{1}{2}
\end{array}$$

- Heun Method: Update of x in three steps

$$\begin{aligned}
k_1 &= f(x(t), u_d), \\
k_2 &= f(x(t) + \Delta k_1, u_d), \\
x(t+\Delta) &= x(t) + \Delta\left(\tfrac{1}{2}k_1 + \tfrac{1}{2}k_2\right).
\end{aligned}$$

Runge-Kutta Methods (in Matlab)

The function `ode23.m` relies on the Butcher tableaus

$$\begin{array}{c|ccc} 0 & & & \\ \frac{1}{2} & \frac{1}{2} & & \\ \frac{3}{4} & 0 & \frac{3}{4} & \\ \hline & \frac{2}{9} & \frac{1}{3} & \frac{4}{9} \end{array} \qquad \text{and} \qquad \begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{3}{4} & 0 & \frac{3}{4} & & \\ 1 & \frac{2}{9} & \frac{1}{3} & \frac{4}{9} & \\ \hline & \frac{7}{24} & \frac{1}{4} & \frac{1}{3} & \frac{1}{8} \end{array}$$

- One scheme is used to approximate $x(t+\Delta)$.
- The second scheme is needed to approximate the error, to select the step size Δ.

The function `ode45.m` relies on the Butcher tableaus

$$\begin{array}{c|ccccccc} 0 & & & & & & & \\ \frac{1}{5} & \frac{1}{5} & & & & & & \\ \frac{3}{10} & \frac{3}{40} & \frac{9}{40} & & & & & \\ \frac{4}{5} & \frac{44}{45} & -\frac{56}{15} & \frac{32}{9} & & & & \\ \frac{8}{9} & \frac{19372}{6561} & -\frac{25360}{2187} & \frac{64448}{6561} & -\frac{212}{729} & & & \\ 1 & \frac{9017}{3168} & -\frac{355}{33} & \frac{46732}{5247} & \frac{49}{176} & -\frac{5103}{18656} & & \\ 1 & \frac{35}{384} & 0 & \frac{500}{1113} & \frac{125}{192} & -\frac{2187}{6784} & \frac{11}{84} & \\ \hline & \frac{35}{384} & 0 & \frac{500}{1113} & \frac{125}{192} & -\frac{2187}{6784} & \frac{11}{84} & 0 \end{array}$$

$$\begin{array}{c|ccccccc} \hline & \frac{5179}{57600} & 0 & \frac{7571}{16695} & \frac{393}{640} & -\frac{92097}{339200} & \frac{187}{2100} & \frac{1}{40} \end{array}$$

Section 3

Stability Notions

Stability Notions

Discrete time systems: Consider

$$x^+ = F(x), \qquad x(0) = x_0 \in \mathbb{R}^n$$

Definition

Consider the origin of the discrete time system.

1. (Stability) The origin is *Lyapunov stable* (or simply *stable*) if, for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x(0)| \leq \delta$ then, for all $k \geq 0$,
$$|x(k)| \leq \varepsilon.$$
2. (Instability) The origin is *unstable* if it is not stable.
3. (Attractivity) The origin is *attractive* if there exists $\delta > 0$ such that if $|x(0)| < \delta$ then
$$\lim_{k\to\infty} x(k) = 0.$$
4. (Asymptotic stability) The origin is *asymptotically stable* if it is both stable and attractive.

Continuous time systems: Consider

$$\dot{x} = f(x), \qquad x(0) = x_0 \in \mathbb{R}^n$$

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Stability Notions (2)

Discrete time systems: Consider

$$x^+ = F(x), \qquad x(0) = x_0 \in \mathbb{R}^n$$

Definition ($\mathcal{KL}$-stability)

The origin of the discrete time system is is globally asymptotically stable, or alternatively $\mathcal{KL}$-stable, if there exists $\beta \in \mathcal{KL}$ such that

$$|x(k)| \leq \beta(|x(0)|, k), \qquad \forall\, k \in \mathbb{N},$$

is satisfied for all $x(0) \in \mathbb{R}^n$.

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Definition (Exponential stability)

Consider the origin of the discrete time system. If there exist $M > 0$ and $\gamma \in (0, 1)$ such that for each $x(0) \in \mathbb{R}^n$ the inequality

$$|x(k)| \leq M|x(0)|\gamma^k, \qquad \forall\, k \in \mathbb{N},$$

is satisfied, then the origin is globally exponentially stable.

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Definition (Exponential stability)

Consider the origin of the discrete time system. If there exist $M > 0$ and $\lambda > 0$ such that for each $x(0) \in \mathbb{R}^n$ the inequality

$$|x(t)| \leq M|x(0)|e^{-\lambda t}, \qquad \forall\, t \in \mathbb{R}_{\geq 0},$$

is satisfied, then the origin is globally exponentially stable.

Lyapunov Characterizations

Consider $x^+ = f(x)$, $0 = f(0)$, $0 \in \mathcal{D} \subset \mathbb{R}^n$ open.

Theorem (Lyapunov stability theorem)

Suppose there exists a continuous function $V : \mathcal{D} \to \mathbb{R}_{\geq 0}$ and functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that, for all $x \in \mathcal{D}$,

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \quad (1)$$
$$V(f(x)) - V(x) \leq 0$$

Then the origin is stable.

Note that

- Decrease condition $V(x^+) = V(f(x)) \leq V(x)$
- differentiability of V (or even continuity) is not required

Consider $\dot{x} = f(x)$, $0 = f(0)$, $0 \in \mathcal{D} \subset \mathbb{R}^n$ open.

Theorem (Lyapunov stability theorem)

Suppose there exists a smooth function $V : \mathcal{D} \to \mathbb{R}_{\geq 0}$ and functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that, for all $x \in \mathcal{D}$,

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \quad (2)$$
$$\langle \nabla V(x), f(x) \rangle \leq 0$$

Then the origin is stable.

Lyapunov Characterizations

Consider $x^+ = f(x)$, $0 = f(0)$, $0 \in \mathcal{D} \subset \mathbb{R}^n$ open.

Theorem (Lyapunov stability theorem)

Suppose there exists a continuous function $V : \mathcal{D} \to \mathbb{R}_{\geq 0}$ and functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that, for all $x \in \mathcal{D}$,

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \tag{1}$$

$$V(f(x)) - V(x) \leq 0$$

Then the origin is stable.

Note that

- Decrease condition $V(x^+) = V(f(x)) \leq V(x)$
- differentiability of V (or even continuity) is not required

Theorem (Asymptotic stability)

Suppose there exists a continuous function $V : \mathcal{D} \to \mathbb{R}_{\geq 0}$, and functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\rho \in \mathcal{P}$ satisfying $\rho(s) < s$ for all $s > 0$, such that, for all $x \in \mathcal{D}$, (1) holds and

$$V(f(x)) - V(x) \leq -\rho(V(x)).$$

Then the origin is asymptotically stable.

Consider $\dot{x} = f(x)$, $0 = f(0)$, $0 \in \mathcal{D} \subset \mathbb{R}^n$ open.

Theorem (Lyapunov stability theorem)

Suppose there exists a smooth function $V : \mathcal{D} \to \mathbb{R}_{\geq 0}$ and functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that, for all $x \in \mathcal{D}$,

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \tag{2}$$

$$\langle \nabla V(x), f(x) \rangle \leq 0$$

Then the origin is stable.

Theorem (Asymptotic stability)

Suppose there exists a smooth function $V : \mathcal{D} \to \mathbb{R}_{\geq 0}$, and functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\rho \in \mathcal{P}$, such that, for all $x \in \mathcal{D}$, (2) holds and

$$\langle \nabla V(x), f(x) \rangle \leq -\rho(V(x)).$$

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Lyapunov Characterizations (2)

Consider $x^+ = f(x)$, $0 = f(0)$, $0 \in \mathcal{D} \subset \mathbb{R}^n$ open.

Theorem (Exponential stability)

Suppose there exists a continuous function $V : \mathcal{D} \to \mathbb{R}_{\geq 0}$ and constants $\lambda_1, \lambda_2 > 0$, $p \geq 1$, and $c \in (0,1)$ such that, for all $x \in \mathcal{D}$

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$$V(f(x)) - V(x) \leq -cV(x).$$

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then the origin is uniformly globally asymptotically stable.

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Linear systems

Consider the discrete time linear system

$$x^+ = Ax, \qquad x(0) \in \mathbb{R}^n \qquad \text{[Solution } x(k) = A^k x(0)\text{]}$$

Theorem

The following properties are equivalent:

1. *The origin $x^e = 0$ is exponentially stable;*
2. *The eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ of A satisfy $|\lambda_i| < 1$ for all $i = 1, \dots, n$; and*
3. *For $Q \in \mathcal{S}^n_{>0}$ there exists a unique $P \in \mathcal{S}^n_{>0}$ satisfying the discrete time Lyapunov equation*

$$A^T PA - P = -Q.$$

A matrix A which satisfies $|\lambda_i| < 1$ for all $i = 1, \dots, n$ is called a *Schur matrix*.

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Stability Preservation of Discretized Systems

Consider (continuous time system):

$$\dot{x} = \lambda x, \qquad \lambda \in \mathbb{R}$$

Euler discretization ($\Delta > 0$):

$$x^+ = x + \Delta\lambda x = (1 + \Delta\lambda)x$$

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Note that

- we have only considered the Euler method and linear systems

$\rightsquigarrow$ See sections on 'stability' in references on 'numerical solution of differential equations'

Section 4

Controllability and Observability

Controllability and Observability

Consider

$$x^+ = Ax + Bu, \qquad y = Cx + Du.$$

Definition (Controllability)

The pair (A, B) is said to be controllable, if for all $x_1, x_2 \in \mathbb{R}^n$ there exists $K \in \mathbb{N}$ and $u : \mathbb{N}_0 \to \mathbb{R}^m$ such that

$$x_2 = A^K x_1 + \sum_{i=1}^{K} A^{K-i} Bu(i-1).$$

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The pair (A, C) is said to be observable, if for all $x_1, x_2 \in \mathbb{R}^n$, $x_1 \neq x_2$ there exists $K \in N$ such that

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Controllability:

- Kalman matrix: $\mathrm{rank}\left([B \; AB \; A^2B \; \cdots \; A^{n-1}B]\right) = n$
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Note that:

- Different to the continuous time setting, K cannot be chosen arbitrarily small.

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Example:

- Consider the controllable pair

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

  Consider the states $x_1 = [0, 0, 1]^T$ and $x_2 = [0, 0, 0]^T$. Then it holds that

$$Ax_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad A^2x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad A^3x_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Hence, without input, the origin is reached in $K = n = 3$ steps $x_2 = A^3 x_1$.
- Due to the vector B which is only unequal to zero in the last entry, x_1 cannot be steered to the origin in fewer steps.

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Loss of Controllability:

- Consider $\dot{x} = A_c x + B_c u$:

$$A_c = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

- Exact discretization:

$$A_{de}(\Delta) = e^{A_c \Delta} = \begin{bmatrix} \cos(\Delta) & \sin(\Delta) \\ -\sin(\Delta) & \cos(\Delta) \end{bmatrix}$$

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Lemma

Consider the pair (A, B) and let (A_{de}, B_{de}) be defined through exact discretization for $\Delta > 0$. The pair (A_{de}, B_{de}) is controllable if and only if $(e^{A\Delta}, B)$ is controllable and A has no eigenvalues of the form $\frac{2}{\Delta}\pi\ell$, $\ell \in \mathbb{N}$.

Introduction to Nonlinear Control

Stability, control design, and estimation

Philipp Braun & Christopher M. Kellett
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Part I:

Chapter 5: Discrete Time Systems

- 5.1 Discrete Time Systems – Fundamentals
- 5.2 Sampling From Continuous to Discrete Time
- 5.3 Stability Notions
- 5.4 Controllability and Observability

Australian National University