Introduction to Nonlinear Control

Stability, control design, and estimation

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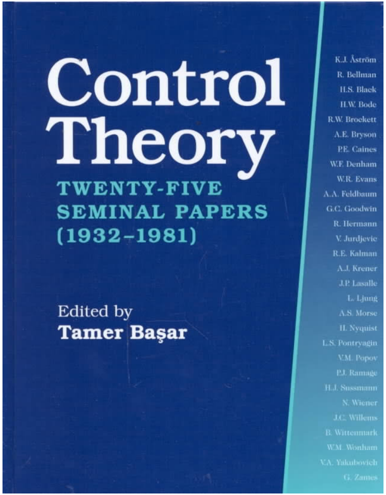
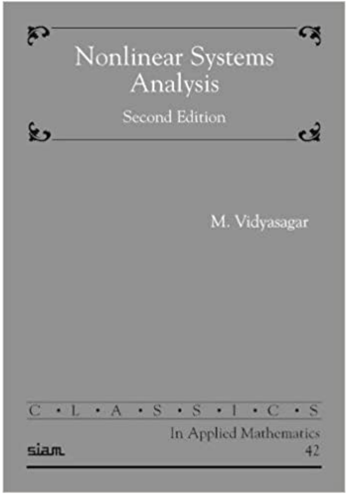
Part I:

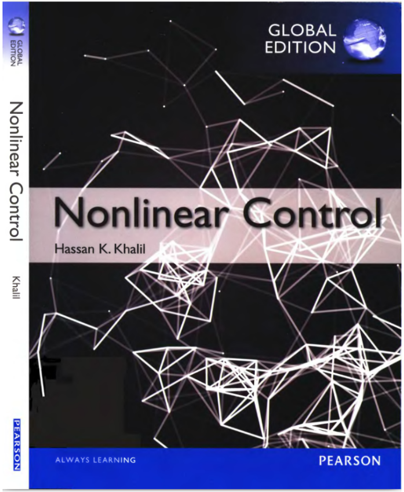
Chapter 6: Absolute Stability

Australian National University

Absolute Stability







Absolute Stability

Section 1

A Commonly Ignored Design Issue

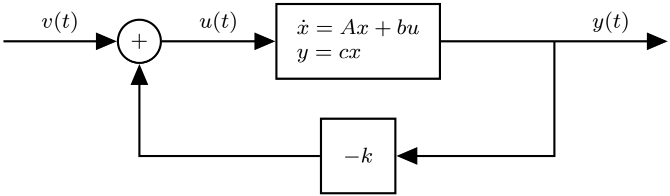
A Commonly Ignored Design Issue

Linear system: $(A \in \mathbb{R}^{n\times n}, b \in \mathbb{R}^{n\times 1}, c \in \mathbb{R}^{1\times n})$

$$\dot{x} = Ax + bu, \qquad y = cx,$$

Feedback interconnection: $u = -ky$

$$\dot{x} = (A - bkc)x,$$



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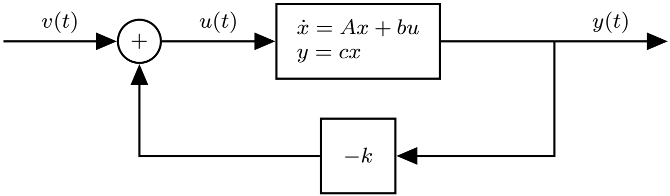

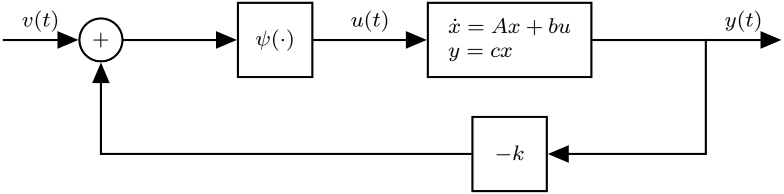

A Commonly Ignored Design Issue (2, Example: Pendulum)

Consider:

- Linearization of the inverted pendulum in the upright position

$$A = \begin{bmatrix} 0 & 1 \\ 9.81 & -0.1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad c = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

  (for given parameters)
- We know that the closed loop system $\dot{x} = A - bkc$, $k = 10$ is asymptotically stable (i.e., $A - bkc$ is Hurwitz)

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- We know that the closed loop system $\dot{x} = A - bkc$, $k = 10$ is asymptotically stable (i.e., $A - bkc$ is Hurwitz)

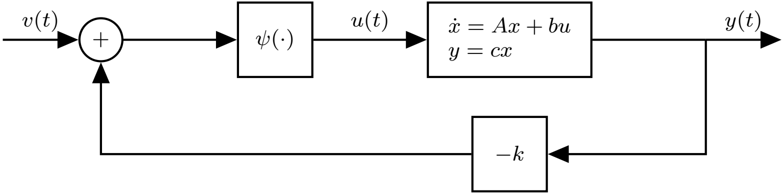
However:

- Any motor used to drive the cart has limited power $u \in [u_{lb}, u_{ub}]$, $u_{lb}, u_{ub} \in \mathbb{R}$.
- Hence,

$$\psi(e) = \begin{cases} u_{lb}, & \text{for } e \leq u_{lb}, \\ e, & \text{for } u_{lb} \leq e \leq u_{ub}, \\ u_{ub}, & \text{for } e \geq u_{ub}, \end{cases}$$

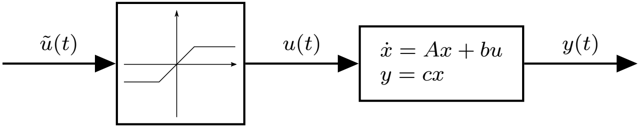
  (where $e(t) = v(t) - ky(t)$ denotes the error variable)

- **Question:** Is the origin of $\dot{x} = Ax - b\psi(kcx)$ asymptotically stable?



A Commonly Ignored Design Issue (3, Saturations)

Input saturation block diagram:

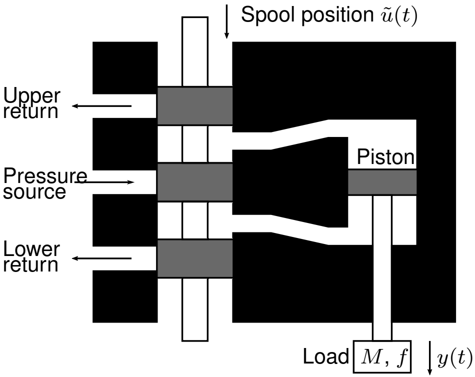


The *saturation function* $\mathrm{sat} : \mathbb{R} \to [-1, 1]$:

$$\mathrm{sat}(y) = \begin{cases} -1, & \text{for } y \leq -1, \\ y, & \text{for } -1 \leq y \leq 1, \\ 1, & \text{for } y \geq 1. \end{cases}$$

- From the normalized function a specific saturation can be obtained through an appropriate scaling and translation.

A Commonly Ignored Design Issue (4, Example: A servo-valve)



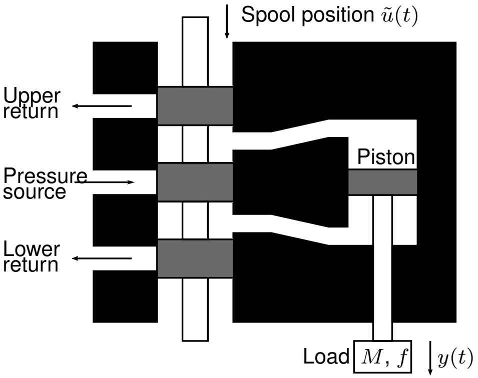
- The linear dynamics are defined through the matrices

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{B}{M} \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} \frac{K}{M} & 0 \end{bmatrix},$$

where $\quad K = a\frac{\frac{\partial g}{\partial x}}{\frac{\partial g}{\partial P}} \quad$ and $\quad B = f + \frac{a^2}{\frac{\partial g}{\partial P}}.$

- Here, $g(x, P)$ denotes the flow, a the area of the piston, P the pressure, and f the viscous friction.

A Commonly Ignored Design Issue (4, Example: A servo-valve)



- Raising the spool allows an inflow of pressure
- Simultaneously pressure drop via the upper return so that the piston will rise
- Note the overlap near the openings

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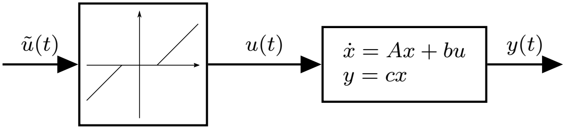

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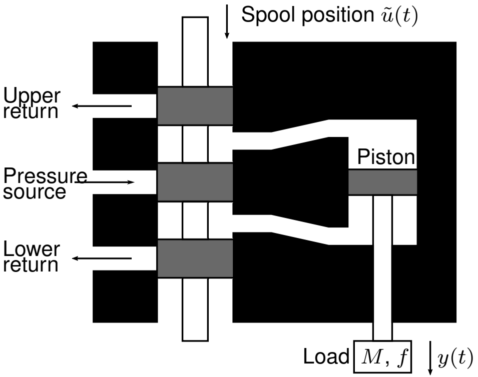

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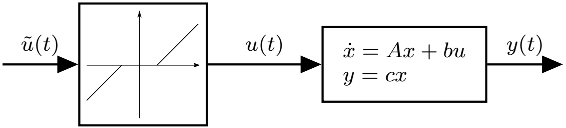
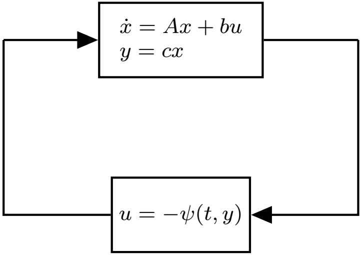


Deadzone $\mathrm{dz} : \mathbb{R} \to \mathbb{R}$:

$$\mathrm{dz}(y) = \begin{cases} y + 1, & \text{for } y \leq -1, \\ 0, & \text{for } -1 \leq y \leq 1, \\ y - 1, & \text{for } y \geq 1. \end{cases}$$

A Commonly Ignored Design Issue (5, The Lur'e Problem)

Consider the feedback interconnection:



Lur'e problem:

- Which conditions on the functions $\psi : \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}$ guarantee asymptotic stability of the origin?

Note that:

- The nonlinearity can be time-dependent
- We assume that the reference signal $v(t)$ is zero.
- While we focus on the SISO case, many results can be extended to the MIMO case.

A Commonly Ignored Design Issue (5, The Lur'e Problem)

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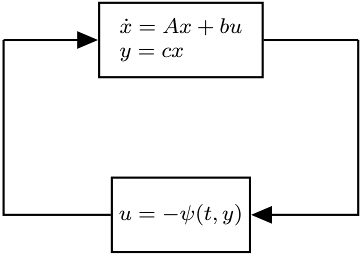

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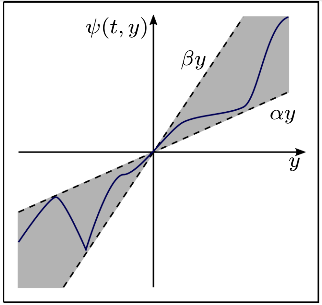
- While we focus on the SISO case, many results can be extended to the MIMO case.

Definition (Sector condition)

Let $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, and $\Omega \subset \mathbb{R}$. A nonlinearity $\psi : \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}$ satisfies a sector condition if

$$\alpha y^2 \leq y\psi(t, y) \leq \beta y^2$$

for all $t \geq 0$ and for all $y \in \Omega$. For $\Omega = \mathbb{R}$ we say that the sector condition is satisfied globally.



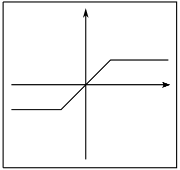
A Commonly Ignored Design Issue (6, The Sector Condition)

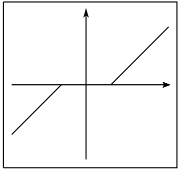
Common nonlinearities: $\mathrm{sign} : \mathbb{R} \to \mathbb{R}$,

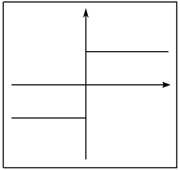
$$\mathrm{sat}(y) = \begin{cases} -1, & \text{for } y \leq -1, \\ y, & \text{for } -1 \leq y \leq 1, \\ 1, & \text{for } y \geq 1. \end{cases}$$

$$\mathrm{dz}(y) = \begin{cases} y+1, & \text{for } y \leq -1, \\ 0, & \text{for } -1 \leq y \leq 1, \\ y-1, & \text{for } y \geq 1. \end{cases}$$

$$\mathrm{sign}(y) = \begin{cases} -1, & \text{for } y < 0, \\ 0, & \text{for } y = 0, \\ 1, & \text{for } y > 0, \end{cases}$$







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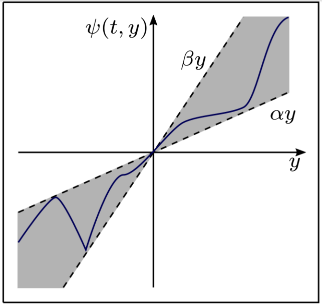

Question:

- Which nonlinearity satisfies a sector condition?

A Commonly Ignored Design Issue (7, Absolute Stability)

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Definition (Absolute stability)

Let $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, and $\Omega \subset \mathbb{R}$. The Lur'e system

$$\dot{x} = Ax - b\psi(t, y)$$

is called absolutely stable (with respect to α, β, Ω) if the origin is asymptotically stable for all $\psi : \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}$ satisfying the sector condition for all $t \geq 0$ and for all $y_0 \in \Omega$.

Historical Perspective on the Lur'e Problem

Conjecture (Aizerman's Conjecture (1949))

Let $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, and suppose the origin of the linear system $\dot{x} = Ax + bu$, $y = cx$ is globally asymptotically stable for all linear feedbacks

$$u = -\psi(y) = -ky, \quad k \in [\alpha, \beta].$$

Then the origin is globally asymptotically stable for all nonlinear feedbacks in the sector

$$\alpha \leq \frac{\psi(y)}{y} \leq \beta, \quad y \neq 0.$$

$\rightsquigarrow$ Conjecture was shown to be wrong through counterexamples.

Conjecture (Kalman's Conjecture (1957))

Let $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, and suppose the origin of the linear system $\dot{x} = Ax + bu$, $y = cx$ is globally asymptotically stable for all linear feedbacks

$$u = -\psi(y) = -ky, \quad k \in [\alpha, \beta].$$

Then the origin is globally asymptotically stable for all nonlinear feedbacks belonging to the incremental sector

$$\alpha \leq \frac{\partial}{\partial y}\psi(y) \leq \beta.$$

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Section 2

Historical Perspective on the Lur'e Problem

Historical Perspective on the Lur'e Problem (Detour into Frequency domain)

Definition (Positive real)

A transfer function $H(s)$ is positive real if

$$\mathrm{Re}(H(s)) \geq 0 \quad \text{for all } \; s \in \overline{\mathbb{C}}_+.$$

The transfer function is strictly positive real (SPR) if $H(s-\varepsilon)$ is positive real for some $\varepsilon > 0$.

Note that the strictly positive real definition above is equivalent to the requirement that

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Example

Consider $H(s) = \frac{1}{s}$. Then

$$\mathrm{Re}(H(s)) = \mathrm{Re}\left(\frac{1}{\sigma+j\omega}\right) = \mathrm{Re}\left(\frac{\sigma-j\omega}{\sigma^2+\omega^2}\right) = \frac{\sigma}{\sigma^2+\omega^2} \geq 0$$

for $\sigma \geq 0$. So $H(s)$ is positive real.
Consider $H(s) = \frac{1}{s+1}$. Then

$$\mathrm{Re}(H(s)) = \mathrm{Re}\left(\frac{1}{\sigma+j\omega+1}\right) = \mathrm{Re}\left(\frac{\sigma+1-j\omega}{(\sigma+1)^2+\omega^2}\right) = \frac{\sigma+1}{\sigma^2+\omega^2} > 0$$

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for $\sigma \geq 0$. So $H(s)$ is strictly positive real.

Lemma

The transfer function $H(s)$ is strictly positive real if and only if $H(s)$ is Hurwitz (i.e., has all its poles in the open left-half complex plane) and

$$\mathrm{Re}(H(j\omega)) > 0, \quad \text{for all } \omega \in \mathbb{R}.$$

Lemma (Kalman-Yakubovich-Popov Lemma)

Let

$$H(s) = c(sI - A)^{-1}b + d$$

be a transfer function where $A \in \mathbb{R}^{n\times n}$ is Hurwitz, (A, b) is controllable, and (A, c) is observable. Then $H(s)$ is strictly positive real if and only if there exist $P \in \mathcal{S}^n_{>0}$, a (row) vector $L \in \mathbb{R}^{1\times n}$, a number $w \in \mathbb{R}$, and positive constant $\varepsilon > 0$ such that

$$A^T P + PA = -L^T L - \varepsilon P,$$
$$Pb = c^T - L^T w,$$
$$w^2 = 2d.$$

Section 3

Sufficient Conditions for Absolute Stability

Sufficient Conditions for Absolute Stability

Theorem (Absolute stability)

Assume that A is Hurwitz, (A, b, c, d) is controllable and observable, $H(s) = c(sI - A)^{-1}b + d$ is strictly positive real, and $\psi(t, y)$ is in the sector $[0, \infty)$. Then the origin of the Lur'e system $\dot{x} = Ax - b\psi(t, y)$ is globally exponentially stable (i.e., the system is absolutely stable).

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- Idea: apply loop transformation to generalize the result

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$$\begin{aligned}\dot{x} &= Ax + b(-\gamma cx + \gamma cx - \psi(t, y)) \\ &= (A - \gamma bc) - b(\psi(t, y) - \gamma y) \\ &= \hat{A}x - b\hat{\psi}(t, y)\end{aligned}$$

Note that:

- Result is only applicable to systems with A Hurwitz
- Idea: apply loop transformation to generalize the result

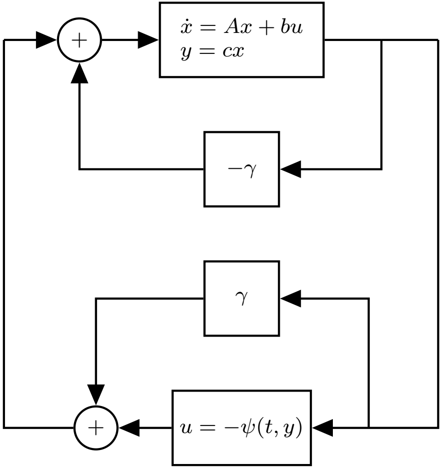
If $\psi(t, y)$ satisfies

$$\alpha y^2 \leq y\psi(t, y) \leq \beta y^2, \qquad \alpha, \beta \in \mathbb{R}$$

then $\hat{\psi}(t, y) = \psi(t, y) - \gamma y$ satisfies the sector condition

$$\hat{\alpha} y^2 \leq y\hat{\psi}(t, y) \leq \hat{\beta} y^2 \qquad \hat{\alpha} = \alpha - \gamma, \hat{\beta} = \beta - \gamma, \quad \gamma \in \mathbb{R}$$

Let $\gamma \in \mathbb{R}$ such that $\hat{A} = A - \gamma bc$ is Hurwitz.
The loop transformation satisfies the assumptions of the Theorem as long as $\hat{\alpha} > 0$.



Circle Criterion

Consider the Lur'e system $\dot{x} = Ax - b\psi(t,y)$ with

$$\alpha y^2 \leq y\psi(t,y) \leq \beta y^2, \qquad \alpha, \beta \in \mathbb{R}$$

Consider the loop transformation

$$\begin{aligned} \dot{x} &= Ax + b(-\gamma cx + \gamma cx - \psi(t,y)) \\ &= (A - \gamma bc) - b(\psi(t,y) - \gamma y) = \hat{A}x - b\hat{\psi}(t,y) \end{aligned}$$

with $\quad 0 \leq y\hat{\psi}(t,y) \leq \hat{\beta} y^2, \qquad \gamma = \alpha, \quad \hat{\beta} \doteq \beta - \alpha$

Lemma

Consider the Lur'e system with (A,b,c) controllable and observable, and transfer function $G(s) = c(sI - A)^{-1}b$. The system (A,b,c) is absolutely stable (with respect to $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, $\Omega = \mathbb{R}$) if $\hat{A} = A - \alpha bc$ is Hurwitz, and

$$H(s) = \frac{1 + \beta G(s)}{1 + \alpha G(s)}$$

is strictly positive real; i.e., if

$$\mathrm{Re}(H(s)) = \mathrm{Re}\left(\frac{1 + \beta G(s)}{1 + \alpha G(s)}\right) > 0 \quad \text{for all} \quad s \in \overline{\mathbb{C}}_+.$$

Circle Criterion (2)

Definitions: (Disc in the complex plane)

- center $\sigma : \mathbb{R}\backslash\{0\} \times \mathbb{R}_{>0} \to \mathbb{R}$
- radius $r : \mathbb{R}\backslash\{0\} \times \mathbb{R}_{>0} \to \mathbb{R}$
- for $\alpha \neq 0$ and $\beta > 0$ we define

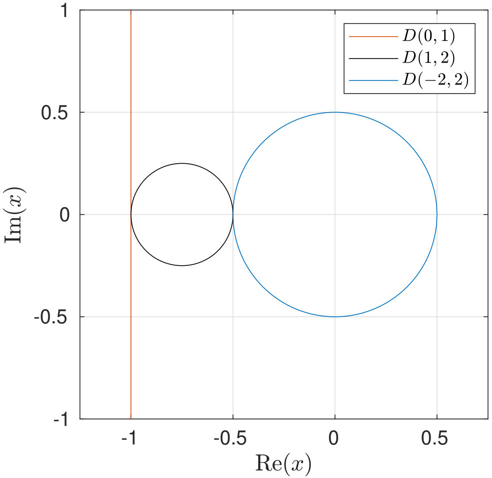
$$\sigma(\alpha,\beta) = \frac{1}{2}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right), \quad r(\alpha,\beta) = \frac{\text{sign}(\alpha)}{2}\left(\frac{1}{\alpha} - \frac{1}{\beta}\right)$$

Then, the disc $D(\cdot,\cdot)$ is defined as

$$D(\alpha,\beta) = \begin{cases} \{x \in \mathbb{C} : x = -\frac{1}{\beta} + j\omega, \omega \in \mathbb{R}\}, & \text{if } \alpha = 0 < \beta, \\ \{x \in \mathbb{C} : |x - \sigma(\alpha,\beta)| = r(\alpha,\beta)\}, & \text{if } 0 < \alpha < \beta, \\ \{x \in \mathbb{C} : |x - \sigma(\alpha,\beta)| = r(\alpha,\beta)\}, & \text{if } \alpha < 0 < \beta. \end{cases}$$

Note that

- for $\alpha \neq 0$, $D(\alpha,\beta)$ defines a disc centered around $\sigma(\alpha,\beta)$ with radius $r(\alpha,\beta)$
- for $\alpha = 0$, $D(0,\beta)$ defines a vertical line



Circle Criterion (3), (First Case)

First case ($\alpha = 0 < \beta$): $H(s)$ reduces to

$$H(s) = 1 + \beta G(s)$$

Then

- $H(s)$ has the same poles as $G(s)$

According to the Lemma

- $G(s)$ (or A) needs to be Hurwitz
- $\mathrm{Re}(1 + \beta G(j\omega)) > 0 \;\; \forall\, \omega \in \mathbb{R}$ needs to be satisfied

Note that

- since $\beta > 0$, the last item is equivalent to

$$\mathrm{Re}(G(j\omega)) > -\tfrac{1}{\beta}, \quad \text{for all } \omega \in \mathbb{R}.$$

- this corresponds to the Nyquist plot of $G(j\omega)$ being to the right of the vertical line in the complex plane through $-1/\beta$.

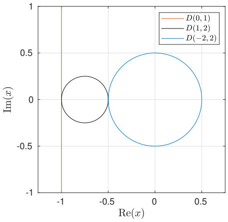
Lemma

Consider the Lur'e system with (A, b, c) controllable and observable, and transfer function $G(s) = c(sI - A)^{-1}b$. The system (A, b, c) is absolutely stable (with respect to $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, $\Omega = \mathbb{R}$) if $\hat{A} = A - \alpha bc$ is Hurwitz, and

$$H(s) = \frac{1 + \beta G(s)}{1 + \alpha G(s)}$$

is strictly positive real; i.e., if

$$\mathrm{Re}(H(s)) = \mathrm{Re}\left(\frac{1 + \beta G(s)}{1 + \alpha G(s)}\right) > 0 \quad \text{for all} \quad s \in \overline{\mathbb{C}}_+.$$



Circle Criterion (3), (First Case)

First case ($\alpha = 0 < \beta$): $H(s)$ reduces to

$$H(s) = 1 + \beta G(s)$$

Then

- $H(s)$ has the same poles as $G(s)$

According to the Lemma

- $G(s)$ (or A) needs to be Hurwitz
- $\mathrm{Re}(1 + \beta G(j\omega)) > 0 \;\; \forall\, \omega \in \mathbb{R}$ needs to be satisfied

Note that

- since $\beta > 0$, the last item is equivalent to

$$\mathrm{Re}(G(j\omega)) > -\tfrac{1}{\beta}, \quad \text{for all } \omega \in \mathbb{R}.$$

- this corresponds to the Nyquist plot of $G(j\omega)$ being to the right of the vertical line in the complex plane through $-1/\beta$.

Recall the disc $D(\cdot,\cdot)$ for $\alpha = 0 < \beta$:

$$D(\alpha,\beta) = \{x \in \mathbb{C} : x = -\tfrac{1}{\beta} + j\omega, \omega \in \mathbb{R}\}$$

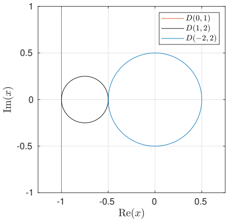
Lemma

Consider the Lur'e system with (A, b, c) controllable and observable, and transfer function $G(s) = c(sI - A)^{-1}b$. The system (A, b, c) is absolutely stable (with respect to $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, $\Omega = \mathbb{R}$) if $\hat{A} = A - \alpha bc$ is Hurwitz, and

$$H(s) = \frac{1 + \beta G(s)}{1 + \alpha G(s)}$$

is strictly positive real; i.e., if

$$\mathrm{Re}(H(s)) = \mathrm{Re}\left(\frac{1 + \beta G(s)}{1 + \alpha G(s)}\right) > 0 \quad \textit{for all} \quad s \in \overline{\mathbb{C}}_+.$$



Circle Criterion (4), (Second Case)

Second case ($0 < \alpha < \beta$): $H(s)$ is strictly positive real if

$$\mathrm{Re}(H(s)) = \mathrm{Re}\left(\frac{1+\beta G(j\omega)}{1+\alpha G(j\omega)}\right) > 0 \quad \text{for all } \omega \in \mathbb{R}$$

Let $q = G(j\hat{\omega})$ and choose the polar coordinates

$$r_1, r_2 > 0 \text{ and } \theta_1, \theta_2 \in (-\pi, \pi) \text{ so that}$$

$$\tfrac{1}{\beta} + q = r_1 e^{j\theta_1} \quad \text{and} \quad \tfrac{1}{\alpha} + q = r_2 e^{j\theta_2}.$$

Then we can rewrite

$$0 < \mathrm{Re}\left(\frac{\frac{1}{\beta} + G(j\omega)}{\frac{1}{\alpha} + G(j\omega)}\right) = \mathrm{Re}\left(\frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}}\right) = \mathrm{Re}\left(\tfrac{r_1}{r_2} e^{j(\theta_1 - \theta_2)}\right)$$

$$= \tfrac{r_1}{r_2} \mathrm{Re}(\cos(\theta_1 - \theta_2) + j\sin(\theta_1 - \theta_2)) = \tfrac{r_1}{r_2}\cos(\theta_1 - \theta_2)$$

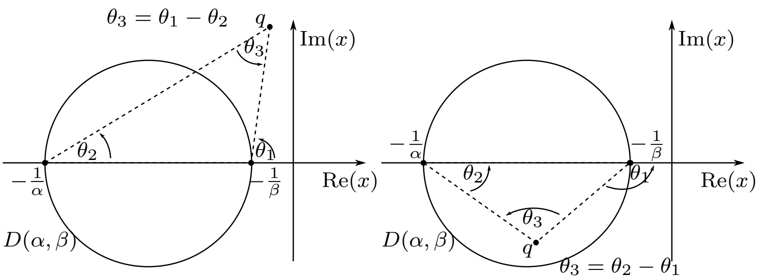
$\rightsquigarrow |\theta_1 - \theta_2| < \frac{\pi}{2}$

$\rightsquigarrow \mathrm{Re}(H(s)) > 0 \ \forall\ \omega \in \mathbb{R} \rightsquigarrow$ Nyquist plot lies entirely outside the disc.

Note that:

- $H(s)$ needs to be Hurwitz
- Poles: $0 = 1 + \alpha G(j\omega)$
- the Nyquist plot needs to encircle the point $-1/\alpha$ as many times as there are right-half plane poles of $G(s)$.

Absolute stability:
The Nyquist plot does not enter the disc and encircles the disc as many times as there are right-half plane poles of $G(s)$.



Circle Criterion (5), (Third Case)

Third case ($\alpha < 0 < \beta$): $H(s)$ is strictly positive real if

$$\mathrm{Re}(H(s)) = \mathrm{Re}\left(\frac{1+\beta G(j\omega)}{1+\alpha G(j\omega)}\right) > 0 \quad \text{for all } \omega \in \mathbb{R}$$

Let $q = G(j\hat{\omega})$ and choose the polar coordinates

$$r_1, r_2 > 0 \text{ and } \theta_1, \theta_2 \in (-\pi, \pi) \text{ so that}$$

$$\frac{1}{\beta} + q = r_1 e^{j\theta_1} \quad \text{and} \quad \frac{1}{\alpha} + q = r_2 e^{j\theta_2}.$$

Then we can rewrite

$$0 > \mathrm{Re}\left(\frac{\frac{1}{\beta} + G(j\omega)}{\frac{1}{\alpha} + G(j\omega)}\right) = \mathrm{Re}\left(\frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}}\right) = \mathrm{Re}\left(\frac{r_1}{r_2} e^{j(\theta_1-\theta_2)}\right)$$

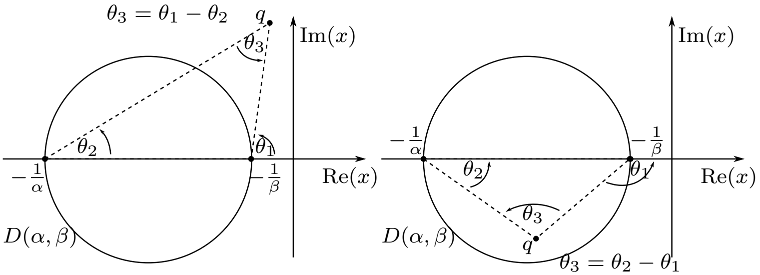
$$= \frac{r_1}{r_2}\mathrm{Re}(\cos(\theta_1-\theta_2) + j\sin(\theta_1-\theta_2)) = \frac{r_1}{r_2}\cos(\theta_1-\theta_2)$$

$\rightsquigarrow |\theta_1 - \theta_2| > \frac{\pi}{2}$
$\rightsquigarrow \mathrm{Re}(H(s)) > 0 \ \forall\, \omega \in \mathbb{R} \rightsquigarrow$ Nyquist plot lies entirely inside the disc.

Note that:

- Since the Nyquist plot cannot leave the disc, it cannot encircle the point $-1/\alpha$
- $G(s)$ cannot have any right-half plane poles, (i.e., $G(s)$ is Hurwitz)

Absolute stability:
The Nyquist plot lies entirely inside the disc.



Circle Criterion (6), (Theorem)

Theorem (Circle Criterion)

Suppose (A, b, c) *is a minimal realization of* $G(s)$ *and* $\psi(t, y)$ *satisfies the sector condition*

$$\alpha y^2 \leq y\psi(t, y) \leq \beta y^2$$

globally. Then the system is absolutely stable if:

1. $\alpha = 0 < \beta$, *the Nyquist plot is to the right of the line* $\mathrm{Re}(s) = -\frac{1}{\beta}$, *(i.e., to the right of* $D(0, \beta)$*) and* $G(s)$ *is Hurwitz;*
2. $0 < \alpha < \beta$, *the Nyquist plot does not enter the disk* $D(\alpha, \beta)$, *and encircles it in the counter-clockwise direction as many times,* N, *as there are right-half plane poles of* $G(s)$*; or*
3. $\alpha < 0 < \beta$, *the Nyquist plot lies in the interior of the disk* $D(\alpha, \beta)$, *and* $G(s)$ *is Hurwitz.*

Circle Criterion (6), (Theorem)

Theorem (Circle Criterion)

Suppose (A, b, c) *is a minimal realization of* $G(s)$ *and* $\psi(t, y)$ *satisfies the sector condition*

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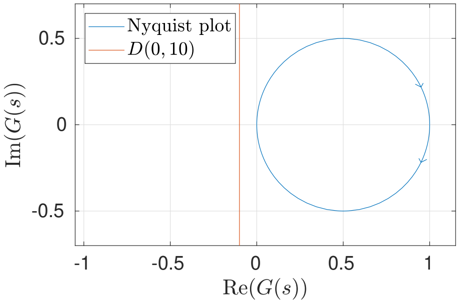
2. $0 < \alpha < \beta$, *the Nyquist plot does not enter the disk* $D(\alpha, \beta)$, *and encircles it in the counter-clockwise direction as many times,* N, *as there are right-half plane poles of* $G(s)$*; or*
3. $\alpha < 0 < \beta$, *the Nyquist plot lies in the interior of the disk* $D(\alpha, \beta)$, *and* $G(s)$ *is Hurwitz.*

Example: Consider

$$G(s) = \frac{1}{s+1} \qquad \text{with pole } s = -1$$

($G(s)$ is Hurwitz, i.e., three items are potentially applicable.)



- Item 1 ($\alpha = 0$ and $\beta = 10$): the Nyquist plot is to the right of the line $D(0, 10)$ $\rightsquigarrow$ $G(s)$ is absolutely stable

Circle Criterion (6), (Theorem)

Theorem (Circle Criterion)

Suppose (A, b, c) *is a minimal realization of* $G(s)$ *and* $\psi(t, y)$ *satisfies the sector condition*

$$\alpha y^2 \leq y\psi(t, y) \leq \beta y^2$$

globally. Then the system is absolutely stable if:

1. $\alpha = 0 < \beta$, *the Nyquist plot is to the right of the line* $\mathrm{Re}(s) = -\frac{1}{\beta}$, *(i.e., to the right of* $D(0, \beta)$*) and* $G(s)$ *is Hurwitz;*
2. $0 < \alpha < \beta$, *the Nyquist plot does not enter the disk* $D(\alpha, \beta)$, *and encircles it in the counter-clockwise direction as many times,* N, *as there are right-half plane poles of* $G(s)$*; or*
3. $\alpha < 0 < \beta$, *the Nyquist plot lies in the interior of the disk* $D(\alpha, \beta)$, *and* $G(s)$ *is Hurwitz.*

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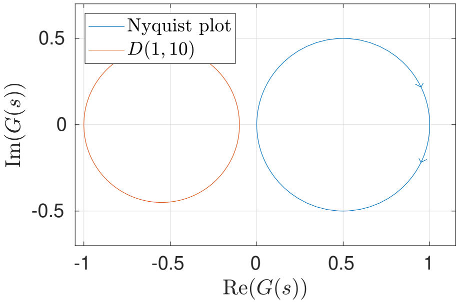

- Item 1 ($\alpha = 0$ and $\beta = 10$): the Nyquist plot is to the right of the line $D(0, 10) \rightsquigarrow G(s)$ is absolutely stable
- Item 2 ($\alpha = 1$ and $\beta = 10$): the Nyquist plot is outside the disc $D(1, 10)$ and encircles it zero times $\rightsquigarrow G(s)$ is absolutely stable

Circle Criterion (6), (Theorem)

Theorem (Circle Criterion)

Suppose (A, b, c) *is a minimal realization of* $G(s)$ *and* $\psi(t, y)$ *satisfies the sector condition*

$$\alpha y^2 \leq y\psi(t, y) \leq \beta y^2$$

globally. Then the system is absolutely stable if:

1. $\alpha = 0 < \beta$, *the Nyquist plot is to the right of the line* $\mathrm{Re}(s) = -\frac{1}{\beta}$, *(i.e., to the right of* $D(0, \beta)$*) and* $G(s)$ *is Hurwitz;*
2. $0 < \alpha < \beta$, *the Nyquist plot does not enter the disk* $D(\alpha, \beta)$, *and encircles it in the counter-clockwise direction as many times,* N, *as there are right-half plane poles of* $G(s)$*; or*
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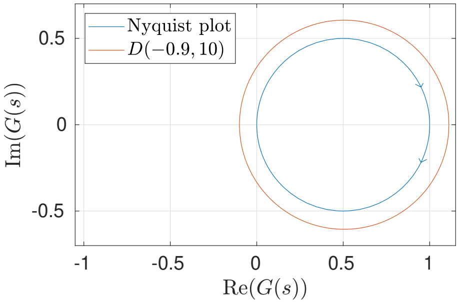


- Item 1 ($\alpha = 0$ and $\beta = 10$): the Nyquist plot is to the right of the line $D(0, 10) \rightsquigarrow G(s)$ is absolutely stable
- Item 2 ($\alpha = 1$ and $\beta = 10$): the Nyquist plot is outside the disc $D(1, 10)$ and encircles it zero times $\rightsquigarrow G(s)$ is absolutely stable
- Item 3 ($\alpha = 1$ and $\beta = 10$): the Nyquist is inside the disc $D(-0.9, 10) \rightsquigarrow G(s)$ is absolutely stable

Circle Criterion (7), (Examples)

Theorem (Circle Criterion)

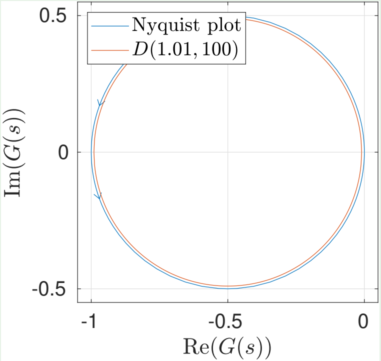
Suppose (A, b, c) *is a minimal realization of* $G(s)$ *and* $\psi(t, y)$ *satisfies the sector condition*

$$\alpha y^2 \leq y\psi(t, y) \leq \beta y^2$$

globally. Then the system is absolutely stable if:

1. $\alpha = 0 < \beta$, *the Nyquist plot is to the right of the line* $\mathrm{Re}(s) = -\frac{1}{\beta}$, *(i.e., to the right of* $D(0, \beta)$*) and* $G(s)$ *is Hurwitz;*
2. $0 < \alpha < \beta$, *the Nyquist plot does not enter the disk* $D(\alpha, \beta)$, *and encircles it in the counter-clockwise direction as many times,* N, *as there are right-half plane poles of* $G(s)$*; or*
3. $\alpha < 0 < \beta$, *the Nyquist plot lies in the interior of the disk* $D(\alpha, \beta)$, *and* $G(s)$ *is Hurwitz.*

Example (Consider $G(s) = \frac{1}{s-1}$)



- $G(s)$ not Hurwitz; one pole in the right-half plane
- $D(1.01, 100)$ encircles the Nyquist plot exactly once in the counter-clockwise direction
- Absolute stability follows (Item 2)

Circle Criterion (8), (Examples)

Theorem (Circle Criterion)

Suppose (A, b, c) *is a minimal realization of* $G(s)$ *and* $\psi(t, y)$ *satisfies the sector condition*

$$\alpha y^2 \leq y\psi(t, y) \leq \beta y^2$$

globally. Then the system is absolutely stable if:

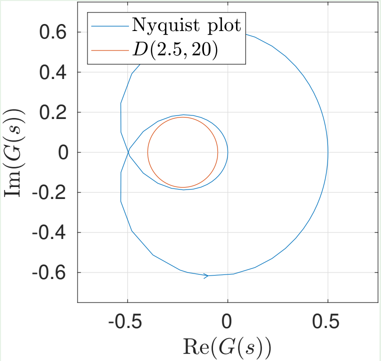
1. $\alpha = 0 < \beta$, *the Nyquist plot is to the right of the line* $\mathrm{Re}(s) = -\frac{1}{\beta}$, *(i.e., to the right of* $D(0, \beta)$*) and* $G(s)$ *is Hurwitz;*
2. $0 < \alpha < \beta$, *the Nyquist plot does not enter the disk* $D(\alpha, \beta)$, *and encircles it in the counter-clockwise direction as many times,* N, *as there are right-half plane poles of* $G(s)$*; or*
3. $\alpha < 0 < \beta$, *the Nyquist plot lies in the interior of the disk* $D(\alpha, \beta)$, *and* $G(s)$ *is Hurwitz.*

Example

Consider the transfer function

$$G(s) = \frac{s+1}{s^2 - 2s + 2} = \frac{s+1}{(s-1+j)(s-1-j)}$$

Two poles in right-half plane $\rightsquigarrow$ absolute stability (Item 2)

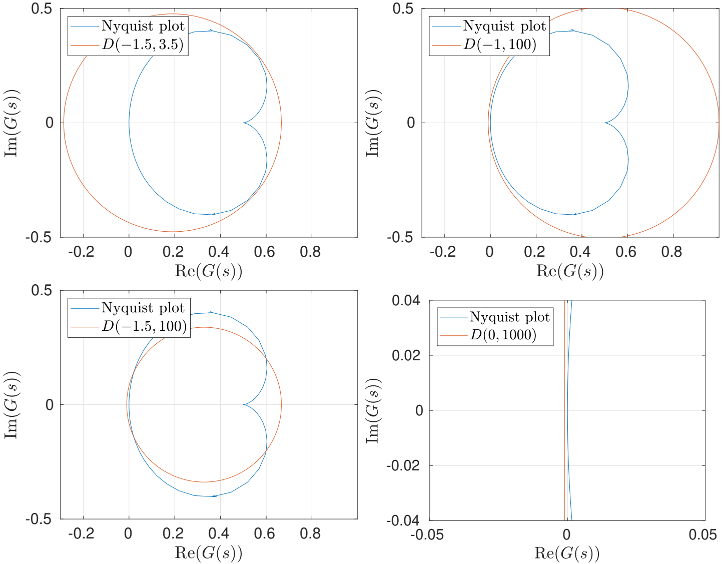


Circle Criterion (9), (Examples)

Consider the transfer function

$$G(s) = \frac{s+1}{s^2+2s+2} = \frac{s+1}{(s+1+j)(s+1-j)}.$$

- $G(s)$ is Hurwitz
- Item 3: Absolute stability for $(\alpha,\beta) = (-1.5, 3.5)$ and for $(\alpha,\beta) = (-1, 100)$ but not for $(\alpha,\beta) = (-1.5, 100)$.
- Item 1: β can be selected arbitrarily large if $\alpha = 0$.



Popov Criterion

Theorem (Popov Criterion)

Suppose A is Hurwitz, (A, b) is controllable, (A, c) is observable, and $\psi(y)$ satisfies the sector condition

$$0 \leq y\psi(y) \leq \beta y^2 \tag{1}$$

for all $y \in \mathbb{R}$. Then the Lur'e system with $G(s) = c(sI - A)^{-1}b$ is absolutely stable if there is an $\eta \geq 0$ with $-\frac{1}{\eta}$ not an eigenvalue of A such that

$$H(s) = 1 + (1 + \eta s)\beta G(s)$$

is strictly positive real.
If ψ only satisfies the sector condition (1) for $y \in \Omega \subset \mathbb{R}$, then the system is absolutely stable with a finite domain.

Note that:

- $G(s)$ needs to be Hurwitz
- $\alpha = 0$
- ψ is memoryless, i.e., $\psi(t, y) = \psi(y)$

Proof.

Assume there exists $\eta \geq 0$ such that $-\frac{1}{\eta}$ is not an eigenvalue of A and $H(s)$ is strictly positive real. Then

$$\begin{aligned} H(s) &= 1 + (1 + \eta s)\beta G(s) \\ &= 1 + \eta\beta cb + c\left(\beta I + \eta\beta A\right)(sI - A)^{-1}b \\ &= d + \hat{c}(sI - A)^{-1}b \end{aligned}$$

where $d = 1 + \eta\beta cb$ and $\hat{c} = c\left(\beta I + \eta\beta A\right)$.
The condition on $-\frac{1}{\eta}$ ensures that $\hat{c} \neq 0$ whenever $c \neq 0$.
Since $H(s)$ is strictly positive real by assumption there exist $P > 0$, L, ε and w satisfying the KYP-equations.
Consider the candidate Lyapunov function

$$V(x) = x^T Px + 2\eta\beta \int_0^y \psi(r)dr.$$

It can be shown that

$$\dot{V}(x) \leq -\varepsilon x^T Px.$$

which shows absolute stability of the system. □

Popov Criterion

Theorem (Popov Criterion)

Suppose A is Hurwitz, (A,b) is controllable, (A,c) is observable, and $\psi(y)$ satisfies the sector condition

$$0 \leq y\psi(y) \leq \beta y^2$$

for all $y \in \mathbb{R}$. Then the Lur'e system with $G(s) = c(sI - A)^{-1}b$ is absolutely stable if there is an $\eta \geq 0$ with $-\frac{1}{\eta}$ not an eigenvalue of A such that

$$H(s) = 1 + (1 + \eta s)\beta G(s)$$

is strictly positive real.

Development of a graphical interpretation:

- Recall: $H(s)$ is strictly positive real if and only if $H(s)$ is Hurwitz and

$$\mathrm{Re}(1 + (1 + j\eta\omega)\beta G(j\omega)) > 0 \quad \text{for all } \omega \in \mathbb{R}.$$

- $H(s)$ has the same poles as $G(s)$ since $-1/\eta$ is not an eigenvalue of A.

- $G(j\omega) \in \mathbb{C}$ can be written as $\gamma + j\delta = G(j\omega)$ for $\gamma, \delta \in \mathbb{R}$ for all $\omega \in \mathbb{R}$. Then

$$\begin{aligned}\mathrm{Re}(1+&\beta G(j\omega) + j\eta\omega\beta G(j\omega)) \\ &= \mathrm{Re}(1 + \beta(\gamma + j\delta) + j\eta\omega\beta(\gamma + j\delta)) \\ &= \mathrm{Re}(1 + \beta\gamma - \eta\omega\beta\delta + j(\beta\delta + \eta\omega\beta\gamma)) \\ &= 1 + \beta\gamma - \eta\omega\beta\delta \\ &= 1 + \beta\,\mathrm{Re}(G(j\omega)) - \eta\omega\beta\,\mathrm{Im}(G(j\omega)).\end{aligned}$$

- Hence, $\mathrm{Re}(H(j\omega)) > 0 \; \forall\, \omega \in \mathbb{R}$ is equivalent to

$$\frac{1}{\beta} + \mathrm{Re}(G(j\omega)) - \eta\omega\,\mathrm{Im}(G(j\omega)) > 0.$$

- If we plot $\mathrm{Re}(G(j\omega))$ versus $\omega\,\mathrm{Im}(G(j\omega))$, the above inequality defines a half space on the right side of the line through $-\frac{1}{\beta}$ of slope $\frac{1}{\eta}$.

- Define the line

$$L(\beta,\eta) = \{x \in \mathbb{C} : x = (-\tfrac{1}{\beta} + j\tfrac{1}{\eta})w, \; w \in \mathbb{R}\}$$

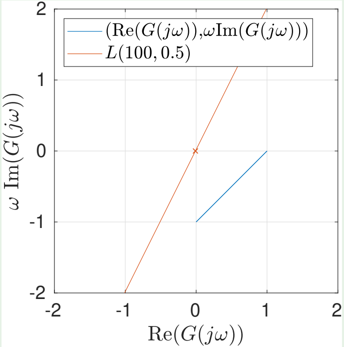
depending on $\beta > 0$ and $\eta \geq 0$.

- We refer to a plot of $\omega\,\mathrm{Im}(G(j\omega))$ versus $\mathrm{Re}(G(j\omega))$ including the line $L(\beta,\eta)$ as a *Popov plot*.

Popov Criterion

Example (Consider $G(s) = \frac{1}{s+1}$)

- $G(s)$ is Hurwitz, i.e., the Popov Criterion is applicable.
- Absolute stability can be concluded for the sector defined through $\alpha = 0$ and $\beta = 100$.



- $G(j\omega) \in \mathbb{C}$ can be written as $\gamma + j\delta = G(j\omega)$ for $\gamma, \delta \in \mathbb{R}$ for all $\omega \in \mathbb{R}$. Then

$$\begin{aligned}\mathrm{Re}(1+&\beta G(j\omega) + j\eta\omega\beta G(j\omega))\\ &= \mathrm{Re}(1 + \beta(\gamma + j\delta) + j\eta\omega\beta(\gamma + j\delta))\\ &= \mathrm{Re}(1 + \beta\gamma - \eta\omega\beta\delta + j(\beta\delta + \eta\omega\beta\gamma))\\ &= 1 + \beta\gamma - \eta\omega\beta\delta\\ &= 1 + \beta\ \mathrm{Re}(G(j\omega)) - \eta\omega\beta\ \mathrm{Im}(G(j\omega)).\end{aligned}$$

- Hence, $\mathrm{Re}(H(j\omega)) > 0\ \forall\ \omega \in \mathbb{R}$ is equivalent to

$$\tfrac{1}{\beta} + \mathrm{Re}(G(j\omega)) - \eta\omega\ \mathrm{Im}(G(j\omega)) > 0.$$

- If we plot $\mathrm{Re}(G(j\omega))$ versus $\omega\,\mathrm{Im}(G(j\omega))$, the above inequality defines a half space on the right side of the line through $-\frac{1}{\beta}$ of slope $\frac{1}{\eta}$.
- Define the line

$$L(\beta,\eta) = \{x \in \mathbb{C} : x = (-\tfrac{1}{\beta} + j\tfrac{1}{\eta})w,\ \ w \in \mathbb{R}\}$$

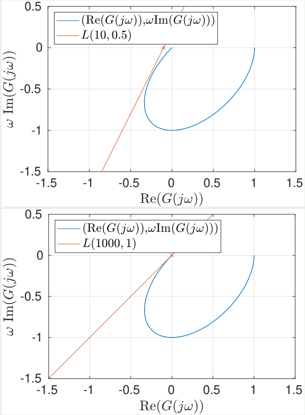
depending on $\beta > 0$ and $\eta \geq 0$.

- We refer to a plot of $\omega\,\mathrm{Im}(G(j\omega))$ versus $\mathrm{Re}(G(j\omega))$ including the line $L(\beta,\eta)$ as a *Popov plot*.

Popov Criterion

Example (Consider $G(s) = \frac{1}{s^2+s+1}$)

- Two (complex) poles in the open left-half plane.
- Absolute stability can be concluded for different values of β (and $\alpha = 0$)



- $G(j\omega) \in \mathbb{C}$ can be written as $\gamma + j\delta = G(j\omega)$ for $\gamma, \delta \in \mathbb{R}$ for all $\omega \in \mathbb{R}$. Then

$$
\begin{aligned}
\mathrm{Re}(1+\beta G(j\omega) + j\eta\omega\beta G(j\omega)) \\
&= \mathrm{Re}(1 + \beta(\gamma + j\delta) + j\eta\omega\beta(\gamma + j\delta)) \\
&= \mathrm{Re}(1 + \beta\gamma - \eta\omega\beta\delta + j(\beta\delta + \eta\omega\beta\gamma)) \\
&= 1 + \beta\gamma - \eta\omega\beta\delta \\
&= 1 + \beta\ \mathrm{Re}(G(j\omega)) - \eta\omega\beta\ \mathrm{Im}(G(j\omega)).
\end{aligned}
$$

- Hence, $\mathrm{Re}(H(j\omega)) > 0\ \forall\ \omega \in \mathbb{R}$ is equivalent to

$$\frac{1}{\beta} + \mathrm{Re}(G(j\omega)) - \eta\omega\ \mathrm{Im}(G(j\omega)) > 0.$$

- If we plot $\mathrm{Re}(G(j\omega))$ versus $\omega\,\mathrm{Im}(G(j\omega))$, the above inequality defines a half space on the right side of the line through $-\frac{1}{\beta}$ of slope $\frac{1}{\eta}$.
- Define the line

$$L(\beta, \eta) = \{x \in \mathbb{C} : x = (-\tfrac{1}{\beta} + j\tfrac{1}{\eta})w,\ w \in \mathbb{R}\}$$

depending on $\beta > 0$ and $\eta \geq 0$.

- We refer to a plot of $\omega\,\mathrm{Im}(G(j\omega)$ versus $\mathrm{Re}(G(j\omega))$ including the line $L(\beta, \eta)$ as a *Popov plot*.

Circle versus Popov Criterion

The Circle Criterion:

- The single-input single-output case discussed here, can be extended to multi-input multi-output systems.
- The Circle Criterion allows for time-varying nonlinearities.
- Checkable conditions for $\alpha \neq 0$ derived from
$$\frac{1+\beta G(s)}{1+\alpha G(s)}.$$
- If $\eta = 0$ and $\alpha = 0$ the Circle Criterion and the Popov Criterion are equivalent.

The Popov Criterion

- Can be extended to the multi-input multi-output settings, but appears to require more structure in the interconnection structure of the input-output behavior and types of nonlinearities that can be accommodated.
- Only applicable to time-invariant nonlinearities.
- Reasonable to assume $G(s)$ that is Hurwitz and take $\alpha = 0$ (due to the factor)
- If $\eta = 0$ and $\alpha = 0$ the Circle Criterion and the Popov Criterion are equivalent.
- The freedom to choose $\eta \geq 0$ can provide a less conservative results.
- The assumption $G(s)$ Hurwitz and $\alpha = 0$ can be accomplished through an appropriate loop transform.

Note that: The Circle Criterion and the Popov Criterion define sufficient conditions.

Introduction to Nonlinear Control

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Part I:

Chapter 6: Absolute Stability

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