Introduction to Nonlinear Control

Stability, control design, and estimation

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Part I:

Chapter 7: Input-to-State Stability

7.1 Motivation & Definition
7.2 Lyapunov Characterization
7.3 System Interconnection
7.4 Integral-to-Integral Estimates and $\mathcal{L}_2$-gain

Australian National University

Input-to-State Stability

1. Motivation & Definition
2. Lyapunov Characterization
3. System Interconnection
   - Casade Interconnection
   - Feedback Interconnection
4. Integral-to-Integral Estimates and $\mathcal{L}_2$-gain
   - System Interconnection

Section 1

Motivation & Definition

Motivation & Definition

Robust Stability:
Consider the linear system

$$\dot{x} = Ax + Ew, \quad x(0) = x_0 \in \mathbb{R}^n,$$

with state x, A Hurwitz, and external disturbance w
Recall the solution $(x(t), t \in \mathbb{R}_{\geq 0})$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Ew(\tau)d\tau$$

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We can calculate/estimate the impact of the disturbance:

$$\begin{aligned} |x(t)| &\leq \left|e^{At}x(0)\right| + \left|\int_0^t e^{A(t-\tau)}Ew(\tau)d\tau\right| \\ &\leq \left\|e^{At}\right\| |x(0)| + \int_0^t \left\|e^{A(t-\tau)}\right\| \|E\| |w(\tau)| d\tau \\ &\leq \left\|e^{At}\right\| |x(0)| + \left(\|E\| \int_0^\infty \left\|e^{A\tau}\right\| d\tau\right) \operatorname*{ess\,sup}_{\tau \geq 0} |w(\tau)| \end{aligned}$$

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This bound consists of two components:

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Input-to-state stability (ISS) for nonlinear systems:

$$\dot{x} = f(x, w), \quad x(0) = x_0 \in \mathbb{R}^n$$

with $w : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$. The set of allowable input functions

$$\mathcal{W} = \{w : \mathbb{R}_{\geq 0} \to \mathbb{R}^m |\ w \text{ essentially bounded}\}.$$

Definition (Input-to-state stability)

The system is said to be *input-to-state stable (ISS)* if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that solutions satisfy

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for all $x \in \mathbb{R}^n$, $w \in \mathcal{W}$, and $t \geq 0$.

- $\gamma \in \mathcal{K}$: *ISS-gain*;
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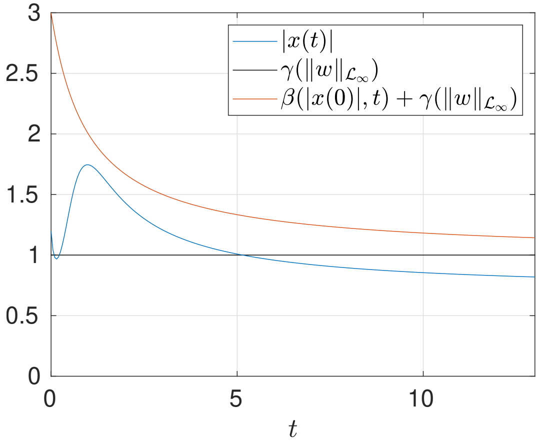

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An equivalent ISS inequality ($\hat{\beta} \in \mathcal{KL}$ and $\hat{\gamma} \in \mathcal{K}$):

$$|x(t)| \leq \max\left\{\hat{\beta}(|x(0)|, t),\ \hat{\gamma}\left(\|w\|_{\mathcal{L}_\infty}\right)\right\}$$

The equivalence follows from

$$a + b \leq \max\{2a, 2b\} \leq 2a + 2b, \quad \forall\ a, b \in \mathbb{R}_{\geq 0}.$$

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Example

Recall that (A Hurwitz)

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satisfies

$$|x(t)| \leq \left\|e^{At}\right\| |x(0)| + \left(\|E\| \int_0^\infty \left\|e^{A\tau}\right\| d\tau\right) \|w\|_{\mathcal{L}_\infty}$$

Then

$$\beta(s, t) \doteq s\|e^{At}\|; \quad \gamma(s) \doteq \left(\|E\| \int_0^\infty \left\|e^{A\tau}\right\| d\tau\right) s,$$

The ISS-gain is linear and the transient bound is given by the product of the identity and an exponentially decaying function of time.

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For linear systems we can conclude that:

- A Hurwitz is sufficient for the system to be ISS.

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Example

Consider the nonlinear/bilinear system:

$$\dot{x} = -x + xw.$$

- The system is 0-input globally asymptotically stable (since $w = 0$ implies $\dot{x} = -x$ and so $x(t) = x(0)e^{-t}$)
- However, consider the bounded input/disturbance $w = 2$. Then $\dot{x} = x$ and so $x(t) = x(0)e^t$.
- Consequently, it is impossible to find $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that

$$|x(t)| = |x(0)|e^t \leq \beta(|x(0)|, t) + \gamma(2).$$

Section 2

Lyapunov Characterization

Lyapunov Characterizations

Definition (Input-to-state stability)

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for all $x \in \mathbb{R}^n$, $w \in \mathcal{W}$, and $t \geq 0$.

Theorem (ISS-Lyapunov function)

$\dot{x} = f(x, w)$ is ISS if and only if there exist a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and $\alpha_1, \alpha_2, \alpha_3, \chi \in \mathcal{K}_\infty$ such that for all $x \in \mathbb{R}^n$ and all $w \in \mathbb{R}^m$

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$
$$|x| \geq \chi(|w|) \quad \Rightarrow \quad \langle \nabla V(x), f(x, w) \rangle \leq -\alpha_3(|x|).$$

Lyapunov Characterizations

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"ISS-Lyapunov function $\Longrightarrow$ ISS":

- First show that $S_w = \{x \in \mathbb{R}^n : \; |x| \leq \chi(|w|)\}$ is forward invariant
- Once solutions enter S_w they remain there $\forall\, t \geq 0$.
- The "size" of this set is dependent only on $|w|$ scaled via $\chi \in \mathcal{K}_\infty$.
- Outside the set S_w, the decrease condition holds
- Apply the comparison principle to obtain a transient bound $\beta \in \mathcal{KL}$.
- Combine S_w and the transient bound to derive

$$|x(t)| \leq \max\left\{\beta(|x(0)|, t), \gamma\left(\|w\|_{\mathcal{L}_\infty}\right)\right\}.$$

$\rightsquigarrow$ The converse direction is significantly more difficult (See the book for a reference)

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$$|x| \geq \chi(|w|) \quad \Rightarrow \quad \langle \nabla V(x), f(x,w)\rangle \leq -\alpha_3(|x|).$$

Further comments:

- The decrease condition is equivalent to ($\sigma \in \mathcal{K}_\infty$)

$$\langle \nabla V(x), f(x,w)\rangle \leq -\alpha_3(|x|) + \sigma(|w|)$$

  (“storage function V with supply pair (α_3, σ)” in some references)
- or (*(exponential) dissipation-form ISS-Lyapunov function*)

$$\langle \nabla V(x), f(x,w)\rangle \leq -V(x) + \sigma(|w|)$$

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$$|x| \geq \chi(|w|) \quad \Rightarrow \quad \langle \nabla V(x), f(x,w)\rangle \leq -V(x)$$

- Note that the functions in the different representations are not the same!

Lyapunov Characterizations (Example & Young's Inequality)

Example

Consider

$$\dot{x} = f(x, w) = -x - x^3 + xw, \quad x(0) = x_0 \in \mathbb{R}$$

The candidate ISS-Lyapunov function $V(x) = \frac{1}{2}x^2$ satisfies

$$\begin{aligned} \langle \nabla V(x), f(x, w) \rangle &= \langle x, -x - x^3 + xw \rangle \\ &= -x^2 - x^4 + x^2 w \end{aligned}$$

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Detour....

Lemma (Young's inequality)

Let $p, q \in \mathbb{R}_{>0}$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then for any $x, y \in \mathbb{R}^n$ the inequality

$$x^T y \leq \frac{1}{p}|x|^p + \frac{1}{q}|y|^q$$

is satisfied.

Application: Let $p = q = 2$, $\varepsilon > 0$, $a, b \in \mathbb{R}^n$. Then

$$a^T b = (\varepsilon a)^T (\tfrac{1}{\varepsilon} b) \leq \frac{\varepsilon^2}{2}|a|^2 + \frac{1}{2\varepsilon^2}|b|^2$$

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Lemma (Young's inequality)

Let $p, q \in \mathbb{R}_{>0}$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$. *Then for any* $x, y \in \mathbb{R}^n$ *the inequality*

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Define $\alpha(s) \doteq s^2 + \frac{1}{2}s^4$ and $\sigma(s) \doteq \frac{1}{2}s^2$, Then

$$\dot{V}(x) \leq -\alpha(|x|) + \sigma(|w|)$$

i.e., V is an ISS-Lyapunov function and the system is ISS.

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Lyapunov Characterizations (Example & Young's Inequality)

Example

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$$\dot{V}(x) \le -\alpha(|x|) + \sigma(|w|)$$

i.e., V is an ISS-Lyapunov function and the system is ISS.

$\rightsquigarrow$ Observe that $\dot{x} = -x - x^3 + xw$ is ISS while $\dot{x} = -x + xw$ is not ISS (even though the linearizations are the same)

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Lyapunov Characterizations (Another Example)

- Consider

$$\begin{aligned} \dot{x}_1 &= -x_1 + w \\ \dot{x}_2 &= -x_2^3 + x_1 x_2 \end{aligned}$$

- Candidate ISS-Lyapunov function

$$V(x) = \tfrac{1}{2}x_1^2 + \tfrac{1}{2}x_2^2.$$

- Then ($\frac{1}{2}|x|^2 \leq V(x) \leq \frac{1}{2}|x|^2$ and)

$$\begin{aligned} \langle \nabla V(x), f(x,w) \rangle &= \left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} -x_1 + w \\ -x_2^3 + x_1 x_2 \end{bmatrix} \right\rangle \\ &= -x_1^2 + x_1 w - x_2^4 + x_2^2 x_1 \\ &\leq -x_1^2 + \tfrac{1}{4}x_1^2 + w^2 - x_2^4 + \tfrac{1}{2}x_2^4 + \tfrac{1}{2}x_1^2 \\ &= -\tfrac{1}{4}x_1^2 - \tfrac{1}{2}x_2^4 + w^2. \end{aligned}$$

[Young's inequality applied to the terms $x_1 w$ and $x_2^2 x_1$.]

- Define

$$\alpha(s) \doteq \begin{cases} \frac{1}{8}s^4, & s \leq 1 \\ \frac{1}{8}s^2, & s > 1 \end{cases} \quad \text{and} \quad \sigma(s) \doteq s^2$$

- Then $\dot{V}(x) \leq -\alpha(|x|) + \sigma(|w|) \rightsquigarrow$ the system is ISS.

Section 3

System Interconnection

System Interconnection

Consider

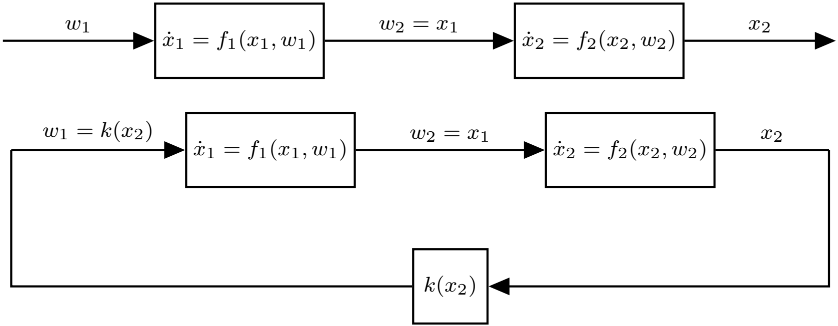
$$\dot{x}_1 = f_1(x_1, w_1)$$
$$\dot{x}_2 = f_2(x_2, w_2)$$

Note that:

- We don't specify the dimensions but assume that the dimensions match in the following!

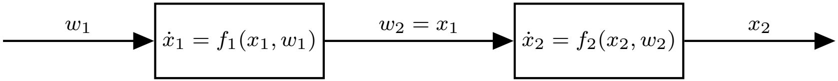
If system 1 and system 2 are ISS

- is the cascade interconnection ISS?
- is the feedback interconnetion ISS?



Cascade Interconnection

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1(x_1, w_1) \\ f_2(x_2, x_1) \end{bmatrix}$$



Definition (Big $\mathcal{O}$ notation)

Consider two positive functions $\rho_1, \rho_2 \in \mathcal{P}$ and let $c \in \mathbb{R}_{\geq 0} \cup \{\infty\}$. We say that $\rho_1(s) = \mathcal{O}[\rho_2(s)]$ as $s \to c$ if and only if

$$\limsup_{s \to c} \left| \frac{\rho_1(s)}{\rho_2(s)} \right| < \infty.$$

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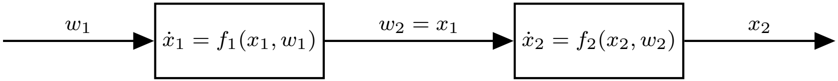

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Example: $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$,

$$\alpha_1(s) = 4s^2 \quad \text{and} \quad \alpha_2(s) = \begin{cases} s^2, & s \leq 1, \\ s^4, & s > 1. \end{cases}$$

Then

$$\limsup_{s \to 0} \left| \frac{\alpha_1(s)}{\alpha_2(s)} \right| = \limsup_{s \to 0} \left| \frac{4s^2}{s^2} \right| = \limsup_{s \to 0} 4 = 4 < \infty$$

i.e., $\alpha_1(s) = \mathcal{O}[\alpha_2(s)]$ as $s \to 0$.

Cascade Interconnection

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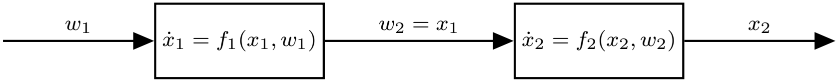

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Cascade Interconnection

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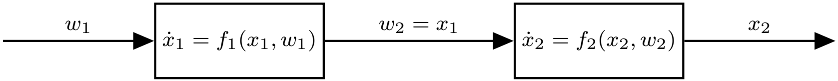

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Theorem (Changing supply pairs)

Consider two systems, $[x_1^T, x_2^T]^T \in \mathbb{R}^n$, with the cascade interconnection $w_2 = x_1$. Assume that $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and $\sigma, \alpha_3 \in \mathcal{K}_\infty$ satisfy

$$\langle \nabla V(x), f(x, w_1) \rangle \leq -\alpha_3(|x|) + \sigma(|w_1|)$$

1. *Suppose that $\tilde{\sigma} \in \mathcal{K}_\infty$ satisfies $\sigma(r) = \mathcal{O}[\tilde{\sigma}(r)]$ as $r \to \infty$. Then there exists $\tilde{\alpha}_3 \in \mathcal{K}_\infty$ so that $(\tilde{\sigma}, \tilde{\alpha}_3)$ satisfy*

   $$\langle \nabla \widetilde{V}(x), f(x, w_1) \rangle \leq -\tilde{\alpha}_3(|x|) + \tilde{\sigma}(|w_1|)$$

   for some $\widetilde{V} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$.

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Cascade Interconnection

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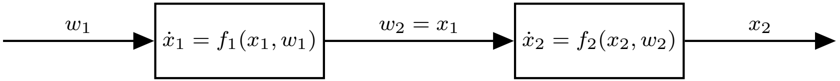

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for some $\widetilde{V} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$.

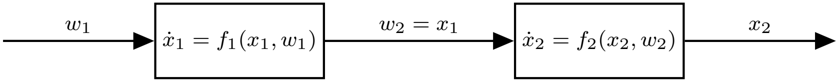
We can freely choose the gain σ for small arguments or we can freely choose the decrease α_3 for large arguments:

$$\limsup_{s\to\infty} \left|\frac{\sigma(s)}{\tilde{\sigma}(s)}\right| < \infty, \quad \text{and} \quad \limsup_{s\to 0} \left|\frac{\tilde{\alpha}_3(s)}{\alpha_3(s)}\right| < \infty.$$

(We cannot modify the gain σ for large arguments or the decrease rate α_3 for small arguments.)

Cascade Interconnection

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1(x_1, w_1) \\ f_2(x_2, x_1) \end{bmatrix}$$



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$$\limsup_{s \to \infty} \left| \frac{\sigma(s)}{\tilde{\sigma}(s)} \right| < \infty, \quad \text{and} \quad \limsup_{s \to 0} \left| \frac{\tilde{\alpha}_3(s)}{\alpha_3(s)} \right| < \infty.$$

(We cannot modify the gain σ for large arguments or the decrease rate α_3 for small arguments.)

Theorem (ISS Cascade)

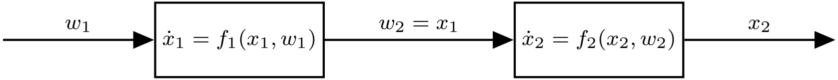
Consider the system with $[x_1, x_2]^T \in \mathbb{R}^n$, $w_2 = x_1$. If each of the subsystems are ISS, then the cascade interconnection is ISS with w_1 as input and x as state.

Proof relies on:

$$\dot{V}_1(x_1) \leq -\alpha_{3,1}(|x_1|) + \sigma_1(|w_1|)$$
$$\dot{V}_2(x_2) \leq -\alpha_{3,2}(|x_2|) + \sigma_2(|w_2|)$$
$$\varphi(s) = \begin{cases} O[\alpha_{3,1}(s)], & \text{as } s \to 0 \\ O[2\sigma_2(s)], & \text{as } s \to \infty \end{cases}$$

Cascade Interconnection

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1(x_1, w_1) \\ f_2(x_2, x_1) \end{bmatrix}$$



Example

Consider

$$\begin{aligned} \dot{x}_1 &= -x_1 + w_1 \\ \dot{x}_2 &= -x_2^3 + x_2 w_2 \end{aligned}$$

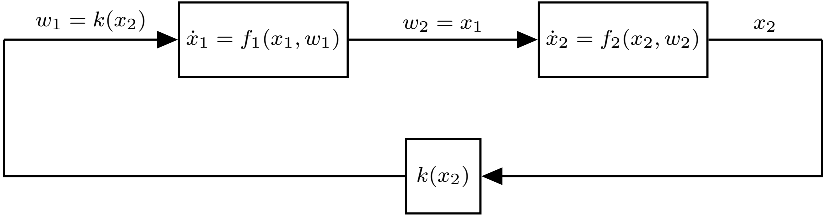
Two Lyapunov functions $V_1(x_1) = \frac{1}{2}x_1^2$ and $V_2(x_2) = \frac{1}{2}x_2^2$ satisfy:

$$\begin{aligned} \dot{V}_1(x_1) &= -x_1^2 + x_1 w_1 \leq -x_1^2 + \tfrac{1}{2}x_1^2 + \tfrac{1}{2}w_1^2 = -\tfrac{1}{2}x_1^2 + \tfrac{1}{2}w_1^2 \\ \dot{V}_2(x_2) &= -x_2^4 + x_2^2 w_2 \leq -x_2^4 + \tfrac{1}{2}x_2^4 + \tfrac{1}{2}w_2^2 = -\tfrac{1}{2}x_2^4 + \tfrac{1}{2}w_2^2 \end{aligned}$$

- $\rightsquigarrow$ The two systems are ISS
- The input and state dimensions match
- $\rightsquigarrow$ The cascade interconnection $w_2 = x_1$ is ISS
- $\rightsquigarrow$ The cascade interconnection $w_1 = x_2$ is ISS

Feedback Interconnection

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1(x_1, k(x_2)) \\ f_2(x_2, x_1) \end{bmatrix}$$



Consider matched ISS-Lyapunov functions satisfying

$$\dot{V}_1(x_1) \leq -\varphi(|x_1|) + \sigma_1(|w_1|)$$
$$\dot{V}_2(x_2) \leq -\alpha_{3,2}(|x_2|) + \varepsilon\varphi(|w_2|), \qquad [\varepsilon \in (0,1)]$$

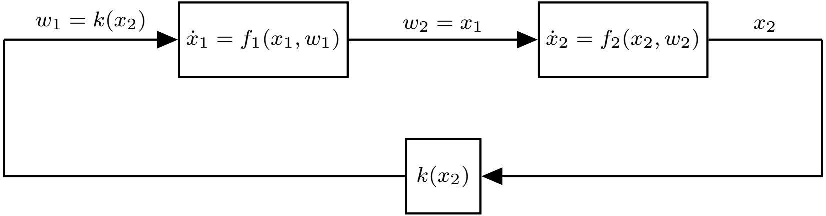
Here, matched refers to

$$\varphi(s) = \begin{cases} \mathcal{O}[\alpha_{3,1}(s)], & \text{as } s \to 0 \\ \mathcal{O}\left[\frac{1}{\varepsilon}\sigma_2(s)\right], & \text{as } s \to \infty \end{cases}$$

Define: $V(x) = V_1(x_1) + V_2(x_2)$. Then

$$\begin{aligned} \dot{V}(x) &= \dot{V}_1(x_1) + \dot{V}_2(x_2) \\ &\leq -\varphi(|x_1|) + \sigma_1(|k(x_2)|) - \alpha_{3,2}(|x_2|) + \varepsilon\varphi(|x_1|) \\ &= -(1-\varepsilon)\varphi(|x_1|) - \alpha_{3,2}(|x_2|) + \sigma_1(|k(x_2)|) \end{aligned}$$

Feedback Interconnection



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Asymptotic stability of the origin?

$$\sigma_1(|k(x_2)|) \leq (1-\bar{\varepsilon})\alpha_{3,2}(|x_2|) \quad \Rightarrow \quad \dot{V}(x(t)) < 0 \ \forall x(t) \neq 0$$

(for $\bar{\varepsilon} \in (0,1)$)

Theorem (ISS small-gain)

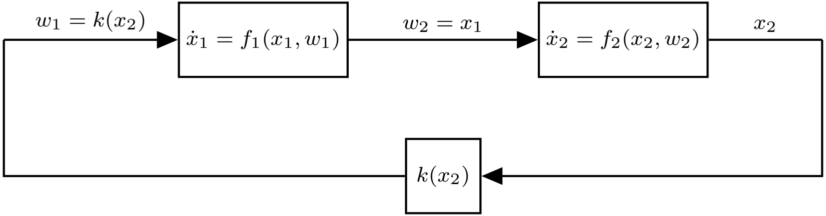
Consider the feedback interconnection. Suppose we have matched ISS-Lyapunov functions for the subsystems. If the nonlinear function $k : \mathbb{R}^{n_1} \to \mathbb{R}^{m_2}$ satisfies

$$|k(x_2)| \leq \sigma_1^{-1}\left((1-\bar{\varepsilon})\alpha_{3,2}(|x_2|)\right)$$

for some $\bar{\varepsilon} \in (0,1)$, then the origin of the closed-loop system is asymptotically stable.

Feedback Interconnection

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1(x_1, k(x_2)) \\ f_2(x_2, x_1) \end{bmatrix}$$



Note that:

- The condition

$$|k(x_2)| \leq \sigma_1^{-1}\left((1-\bar{\varepsilon})\alpha_{3,2}(|x_2|)\right)$$

  is called small-gain condition
- Small-gain theorems place limits on the loop-gain of a feedback system so that signals are not amplified as they traverse the feedback loop.
- Small-gain theorems present sufficient conditions (not necessary conditions)

Feedback Interconnection

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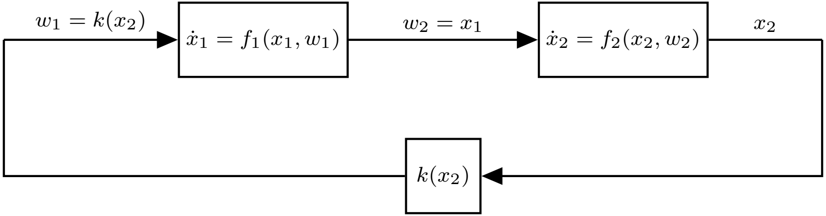

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Theorem

Consider the feedback interconnection with $w_2, x_1 \in \mathbb{R}^{n_1}$ and $w_1, x_2 \in \mathbb{R}^{n_2}$ and $w_1 = k(x_2) = x_2$ and $w_2 = x_1$. If each of the systems is ISS with ISS-Lyapunov functions

$$\dot{V}_1(x_1) \leq -\alpha_{3,1}(V_1(x_1)) + \sigma_1(V_2(x_2))$$

$$\dot{V}_2(x_2) \leq -\alpha_{3,2}(V_2(x_2)) + \sigma_2(V_1(x_1))$$

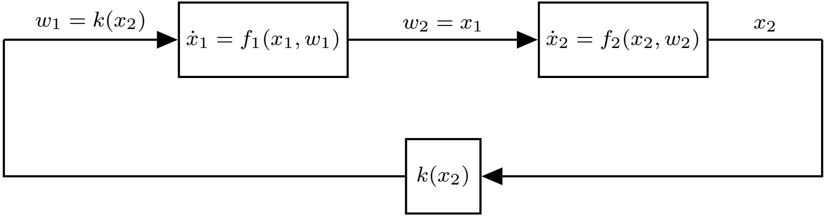
(and $\alpha_{3,1}, \alpha_{3,2}, \sigma_1, \sigma_2 \in \mathcal{K}_\infty$) and if, for all $s \geq 0$,

$$\alpha_{3,1}^{-1} \circ \sigma_2(s) < s, \qquad \alpha_{3,2}^{-1} \circ \sigma_1(s) < s$$

then the origin of the feedback interconnection is asymptotically stable.

Feedback Interconnection

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1(x_1, k(x_2)) \\ f_2(x_2, x_1) \end{bmatrix}$$



Example: Consider

$$\dot{x}_1 = -x_1 + w_1$$
$$\dot{x}_2 = -x_2^3 + x_2 w_2$$

Consider $V_1(x_1) = \frac{\varepsilon}{2} x_1^2$ for $\varepsilon \in (0,1)$ and $V_2(x_2) = \frac{1}{2} x_2^2$. With $\alpha_{3,1}(s) = \frac{\varepsilon}{2} s^4$, $\sigma_2(s) = \frac{1}{2} s^2$ and $\varphi(s) = \frac{1}{2} s^2$ it holds that (verify!)

$$\dot{V}_2(x_2) \leq -\varphi(|x_2|) + \sigma_2(|w_2|)$$
$$\dot{V}_1(x_1) \leq -\alpha_{3,1}(|x_1|) + \varepsilon\varphi(|w_1|)$$

Feedback interconnection $w_2 = k(x_1)$, $w_1 = x_2$

To conclude asymptotic stability the condition

$$|k(x_1)| \leq \sqrt{2(1-\bar{\varepsilon})\frac{\varepsilon}{2}|x_1|^4} = \sqrt{(1-\bar{\varepsilon})\varepsilon} x_1^2$$

needs to be satisfied. (Here $\sigma_2^{-1}(s) = \sqrt{2s}$.)

Since in this example the selection of $\varepsilon \in (0,1)$ in the ISS-Lyapunov function as well as $\bar{\varepsilon} \in (0,1)$ in the Theorem are arbitrary, all of the feedback functions

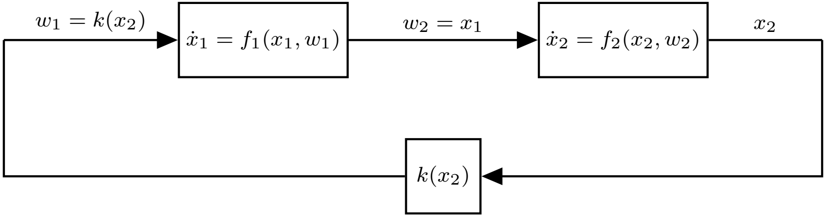
$$k(x_1) = \tfrac{1}{2} x_1^2, \quad k(x_1) = \tfrac{1}{2} \operatorname{sign}(x_1) x_1^2, \quad k(x_1) = \tfrac{1}{2} \operatorname{sat}(x_1^2)$$

satisfy the condition for $\varepsilon = \bar{\varepsilon} = \frac{1}{2}$.

Note that: For $w_1 = k(x_2)$, $w_2 = x_1$, the Lyapunov functions V_1 and V_2 are not matched.

Feedback Interconnection

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1(x_1, k(x_2)) \\ f_2(x_2, x_1) \end{bmatrix}$$



Example: Consider the dynamical system

$$\dot{x}_1 = -x_1^3 + x_1 w_1,$$
$$\dot{x}_2 = -x_2 + \tfrac{1}{2} w_2^2.$$

The functions $V_1(x_1) = \frac{1}{2}x_1^2$ and $V_2(x_2) = \frac{1}{2}x_2^2$ satisfy the estimates

$$\dot{V}_1(x_1) = -x_1^4 + x_1^2 w_1 \leq -x_1^4 + \tfrac{1}{2}x_1^4 + \tfrac{1}{2}w_1^2 = -2V_1(x_1)^2 + V_2(w_1),$$
$$\dot{V}_2(x_2) = -x_2^2 + \tfrac{1}{2}x_2 w_2^2 \leq -x_2^2 + \tfrac{1}{4}x_2^2 + \tfrac{1}{4}w_2^4 = -\tfrac{3}{2}V_2(x_2) + V_1(w_1)^2.$$

Define

$$\alpha_{3,1}(s) = 2s^2, \quad \sigma_1(s) = \frac{1}{2}s, \quad \alpha_{3,2}(s) = \frac{3}{2}s, \quad \sigma_2(s) = s^2$$

It holds that

$$\dot{V}_1(x_1) \leq -\alpha_{3,1}(V_1(x_1)) + \sigma_1(V_2(x_2))$$
$$\dot{V}_2(x_2) \leq -\alpha_{3,2}(V_2(x_2)) + \sigma_2(V_1(x_1))$$

and

$$\alpha_{3,1}^{-1} \circ \sigma_2(s) < s$$
$$\alpha_{3,2}^{-1} \circ \sigma_1(s) < s$$

$\rightsquigarrow$ The origin of the feedback interconnection ($w_1 = x_2$, $w_2 = x_1$) is asymptotically stable.

Section 4

Integral-to-Integral Estimates and $\mathcal{L}_2$-gain

Integral-to-Integral Estimates and $\mathcal{L}_2$-gain

Derivation of an alternate ISS estimate:

- Recall: Dissipation-form ISS-Lyapunov function

$$\begin{aligned}\tfrac{d}{dt}V(x(t)) &= \langle \nabla V(x(t)), f(x(t), w(t))\rangle \\ &\leq -\alpha_3(|x(t)|) + \sigma(|w(t)|)\end{aligned}$$

Integral-to-Integral Estimates and $\mathcal{L}_2$-gain

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- Integration

$$\begin{aligned}V(x(t)) - V(x(0)) \leq &-\int_0^t \alpha_3(|x(\tau)|)d\tau \\ &+\int_0^t \sigma(|w(\tau)|)d\tau.\end{aligned}$$

- Rearrange terms (and $V(x) \leq \alpha_2(|x|)$):

$$\begin{aligned}\int_0^t \alpha_3(|x(\tau)|)d\tau &\leq \int_0^t \alpha_3(|x(\tau)|)d\tau + V(x(t)) \\ &\leq V(x(0)) + \int_0^t \sigma(|w(\tau)|)d\tau \\ &\leq \alpha_2(|x(0)|) + \int_0^t \sigma(|w(\tau)|)d\tau \qquad (1)\end{aligned}$$

Integral-to-Integral Estimates and $\mathcal{L}_2$-gain

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- Rearrange terms (and $V(x) \leq \alpha_2(|x|)$):

$$\begin{aligned}\int_0^t \alpha_3(|x(\tau)|)d\tau &\leq \int_0^t \alpha_3(|x(\tau)|)d\tau + V(x(t)) \\ &\leq V(x(0)) + \int_0^t \sigma(|w(\tau)|)d\tau \\ &\leq \alpha_2(|x(0)|) + \int_0^t \sigma(|w(\tau)|)d\tau \qquad (1)\end{aligned}$$

Lemma

Consider the nonlinear system $\dot{x} = f(x, w)$. If the system is ISS, then there exist $\alpha_2, \alpha_3, \sigma \in \mathcal{K}_\infty$ such that (1) is satisfied for all $t \geq 0$. Conversely, if $\dot{x} = f(x, w)$ is forward complete and satisfies (1) for $\alpha_2, \alpha_3, \sigma \in \mathcal{K}_\infty$ for all $t \geq 0$, then the system is ISS.

Integral-to-Integral Estimates and $\mathcal{L}_2$-gain

Derivation of an alternate ISS estimate:

- Recall: Dissipation-form ISS-Lyapunov function

$$\frac{d}{dt}V(x(t)) = \langle \nabla V(x(t)), f(x(t), w(t)) \rangle$$
$$\leq -\alpha_3(|x(t)|) + \sigma(|w(t)|)$$

- Integration

$$V(x(t)) - V(x(0)) \leq -\int_0^t \alpha_3(|x(\tau)|)d\tau + \int_0^t \sigma(|w(\tau)|)d\tau.$$

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$$\begin{aligned}\int_0^t \alpha_3(|x(\tau)|)d\tau &\leq \int_0^t \alpha_3(|x(\tau)|)d\tau + V(x(t)) \\ &\leq V(x(0)) + \int_0^t \sigma(|w(\tau)|)d\tau \\ &\leq \alpha_2(|x(0)|) + \int_0^t \sigma(|w(\tau)|)d\tau \qquad (1)\end{aligned}$$

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Consider $\dot{x} = Ax + Ew$; A Hurwitz.
Consider $V(x) = x^T P x$, P positive definite, defined through

$$A^T P + PA = -2I.$$

It holds that (Cauchy-Schwarz and Young's inequality)

$$\begin{aligned}\dot{V}(x) &= x^T A^T P x + w^T E^T P x + x^T P A x + x^T P E w \\ &= -2x^T x + 2x^T P E w \leq -2x^T x + 2|x|\,|w|\,\|PE\| \\ &\leq -2x^T x + x^T x + \|PE\|^2 w^T w = -x^T x + \|PE\|^2 w^T w\end{aligned}$$

Integrate and rearrange

$$\int_0^t |x(\tau)|^2 d\tau \leq \lambda_{\max(P)}|x(0)|^2 + \|PE\|^2 \int_0^t |w(\tau)|^2 d\tau$$

Integral-to-Integral Estimates and $\mathcal{L}_2$-gain (2)

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Integrate and rearrange

$$\int_0^t |x(\tau)|^2 d\tau \leq \lambda_{\max(P)} |x(0)|^2 + \|PE\|^2 \int_0^t |w(\tau)|^2 d\tau$$

With $\alpha_3(s) = s^2$, $\alpha_2(s) = \lambda_{\max} s^2$ and $\sigma(s) = \|PE\|^2 s^2$:

$$\int_0^t \alpha_3(|x(t)|)d\tau \leq \alpha_2(|x(0)|) + \int_0^t \sigma(|w(\tau)|)d\tau$$

Alternatively using the $\mathcal{L}_2$-norm:

$$\|x\|_{\mathcal{L}_2}^2 \leq \lambda_{\max(P)} |x(0)|^2 + \gamma^2 \|w\|_{\mathcal{L}_2}^2$$

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With $\alpha_3(s) = s^2$, $\alpha_2(s) = \lambda_{\max} s^2$ and $\sigma(s) = \|PE\|^2 s^2$:

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Alternatively using the $\mathcal{L}_2$-norm:

$$\|x\|_{\mathcal{L}_2}^2 \leq \lambda_{\max(P)} |x(0)|^2 + \gamma^2 \|w\|_{\mathcal{L}_2}^2$$

Definition ($\mathcal{L}_2$-stability)

The system $\dot{x} = f(x, w)$ is said to be $\mathcal{L}_2$*-stable* or to have *finite* $\mathcal{L}_2$*-gain* if there exist constants $\kappa, \gamma > 0$ so that

$$\|x\|_{\mathcal{L}_2}^2 \leq \kappa |x(0)|^2 + \gamma^2 \|w\|_{\mathcal{L}_2}^2$$

for all $w \in \mathcal{W}$.

Note that:

- It is common to assume $x(0) = 0$ and hence the above definition is frequently written simply as

$$\|x\|_{\mathcal{L}_2}^2 \leq \gamma^2 \|w\|_{\mathcal{L}_2}^2.$$

Integral-to-Integral Estimates and $\mathcal{L}_2$-gain (3)

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Connection between input-output $\mathcal{L}_2$-stability and the Bode Plot for linear systems:

$$\dot{x} = Ax + Ew, \qquad y = Cx$$

and its representation in the frequency domain

$$\hat{y}(s) = G(s)\hat{w}(s), \qquad G(s) = C(sI - A)^{-1}E.$$

Integral-to-Integral Estimates and $\mathcal{L}_2$-gain (3)

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Recall Parseval's relation:

$$\|y\|_{\mathcal{L}_2}^2 = \int_0^\infty |y(\tau)|^2 \, d\tau = \frac{1}{2\pi} \int_{-\infty}^\infty |\hat{y}(jw)|^2 \, dw$$

This can be further rewritten

$$\begin{aligned}
\|y\|_{\mathcal{L}_2}^2 &\leq \frac{1}{2\pi} \int_{-\infty}^\infty |G(jw)|^2 |\hat{w}(j\omega)|^2 \, d\omega \\
&\leq \operatorname*{ess\,sup}_{\omega} |G(j\omega)|^2 \frac{1}{2\pi} \int_{-\infty}^\infty |\hat{w}(j\omega)|^2 \, d\omega \\
&= \|G\|_\infty^2 \int_0^\infty |w(\tau)|^2 \, d\tau \\
&= \|G\|_\infty^2 \|w\|_{\mathcal{L}_2}^2.
\end{aligned}$$

$\rightsquigarrow$ With $\gamma = \|G\|_\infty$ the $\mathcal{L}_2$-gain of a linear system is the peak magnitude of the transfer function and can be read off from the Bode Plot.

Note that:

- The estimate also holds for multi-input, multi-output systems. However, in this case the definition of the $\mathcal{H}_\infty$-norm for multi-input, multi-output systems has to be used.

System Interconnections

We assume that:

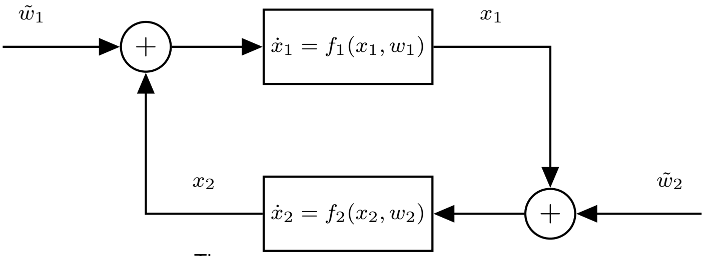
$$x(0) = 0$$

It holds that:

$$\|v_1 + v_2\|^2_{\mathcal{L}_2} \leq \|v_1\|^2_{\mathcal{L}_2} + \|v_2\|^2_{\mathcal{L}_2},$$

Closed loop system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2 + \tilde{w}_1) \\ f_2(x_2, x_1 + \tilde{w}_2) \end{bmatrix}$$



Theorem ($\mathcal{L}_2$ small-gain)

Consider the closed loop system. If each of the subsystems is $\mathcal{L}_2$-stable with gains $\gamma_1, \gamma_2 > 0$, then closed loop system with $w_1 = x_2 + \tilde{w}_1$ and $w_2 = x_1 + \tilde{w}_2$ is $\mathcal{L}_2$-stable if $\gamma_1\gamma_2 < 1$.

Proof: $\mathcal{L}_2$-stability implies

$$\begin{aligned} \|x_1\|^2_{\mathcal{L}_2} &\leq \gamma_1^2 \|w_1\|^2_{\mathcal{L}_2} = \gamma_1^2 \|\tilde{w}_1 + x_2\|^2_{\mathcal{L}_2} \\ &\leq \gamma_1^2 \|\tilde{w}_1\|^2_{\mathcal{L}_2} + \gamma_1^2\gamma_2^2 \|\tilde{w}_2\|^2_{\mathcal{L}_2} + \gamma_1^2\gamma_2^2 \|x_1\|^2_{\mathcal{L}_2} \end{aligned}$$

Then

$$\|x_1\|^2_{\mathcal{L}_2}(1 - \gamma_1^2\gamma_2^2) \leq \gamma_1^2 \|\tilde{w}_1\|^2_{\mathcal{L}_2} + \gamma_1^2\gamma_2^2 \|\tilde{w}_2\|^2_{\mathcal{L}_2}.$$

If $\gamma_1\gamma_2 < 1$ then $\gamma_1^2\gamma_2^2 < 1$ and

$$\|x_1\|^2_{\mathcal{L}_2} \leq \frac{1}{1 - \gamma_1^2\gamma_2^2} \left(\gamma_1^2 \|\tilde{w}_1\|^2_{\mathcal{L}_2} + \gamma_1^2\gamma_2^2 \|\tilde{w}_2\|^2_{\mathcal{L}_2} \right)$$

Same bound on the $\mathcal{L}_2$-norm of x_2 can be derived
The bounds on x_1 and x_2 can be combined as

$$\begin{aligned} \|x\|^2_{\mathcal{L}_2} &= \|x_1\|^2_{\mathcal{L}_2} + \|x_2\|^2_{\mathcal{L}_2} \\ &\leq \tfrac{1}{1-\gamma_1^2\gamma_2^2} \left(\gamma_1^2 \|\tilde{w}_1\|^2_{\mathcal{L}_2} + \gamma_2^2 \|\tilde{w}_2\|^2_{\mathcal{L}_2} + \gamma_1^2\gamma_2^2 (\|\tilde{w}_1\|^2_{\mathcal{L}_2} + \|\tilde{w}_2\|^2_{\mathcal{L}_2}) \right) \end{aligned}$$

Introduction to Nonlinear Control

Stability, control design, and estimation

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Part I:

Chapter 7: Input-to-State Stability

- 7.1 Motivation & Definition
- 7.2 Lyapunov Characterization
- 7.3 System Interconnection
- 7.4 Integral-to-Integral Estimates and $\mathcal{L}_2$-gain

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