Introduction to Nonlinear Control

Stability, control design, and estimation

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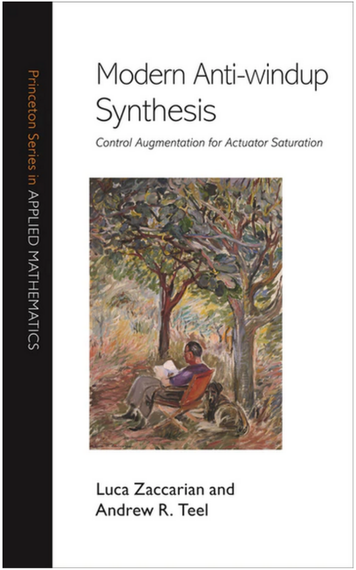
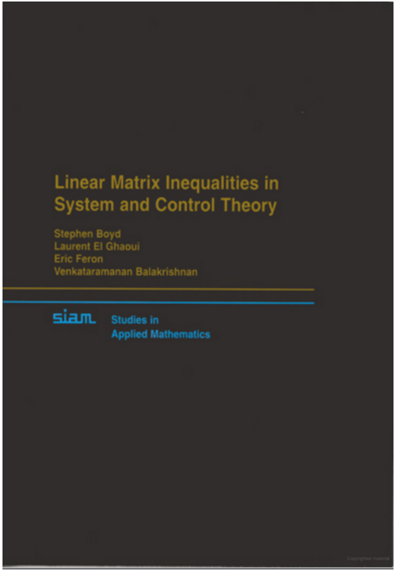
Part II:

Chapter 8: LMI Based Controller and Antiwindup Designs

Australian National University

LMI Based Controller and Antiwindup Designs





LMI Based Controller and Antiwindup Designs

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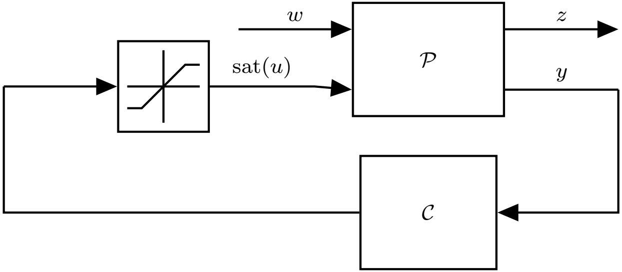


Plant & Controller:

$$\mathcal{P}: \begin{cases} \dot{x}_p = A_p x_p + B_p \operatorname{sat}(u) + B_w w \\ y = C_{p,y} x_p + D_{p,y} w \\ z = C_{p,z} x_p + D_{p,z} w \end{cases}$$

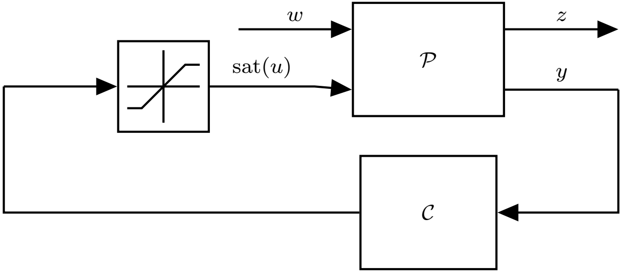
$$\mathcal{C}: \begin{cases} \dot{x}_c = A_c x_c + B_c y \\ u = C_c x_c + D_{c,y} y \end{cases}$$

Compact representation: ($x = [x_p^T, x_c^T]^T \in \mathbb{R}^n$)

$$\left[\begin{array}{c|c|c} A & B & E \\ \hline C & D & F \\ \hline K & L & G \end{array}\right] = \left[\begin{array}{cc|c|c} A_p + B_p D_{c,y} C_{p,y} & B_p C_c & -B_p & B_p D_{c,y} D_{p,y} + B_w \\ B_c C_{p,y} & A_c & 0 & B_c D_{p,y} \\ \hline C_{p,z} & 0 & 0 & D_{p,z} \\ \hline D_{c,y} C_{p,y} & C_c & 0 & D_{c,y} D_{p,y} \end{array}\right]$$

$$\begin{aligned} \dot{x} &= Ax + Bq + Ew \\ z &= Cx + Dq + Fw \\ u &= Kx + Lq + Gw \\ q &= u - \operatorname{sat}(u) \end{aligned}$$

LMI Based Controller and Antiwindup Designs



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Note that:

- The dynamics on the right is more general than the diagram (L, D)
- The system is nonlinear due to the saturation

- If $L \neq 0$, then $u = L(u - \operatorname{sat}(u)) + \mu$ defines an algebraic loop/equation

⇝ Existence and uniqueness of a solution is not automatically satisfied

Well-posedness of algebraic loops

System of interest:

$$
\begin{aligned}
\dot{x} &= Ax + Bq + Ew \\
z &= Cx + Dq + Fw \\
u &= Kx + Lq + Gw \\
q &= u - \mathrm{sat}(u)
\end{aligned}
$$

Algebraic loop: $(\mu = Kx + Gw)$

$$u = L(u - \mathrm{sat}(u)) + \mu$$

Definition (Well-posed algebraic loop)

For $L \in \mathbb{R}^{n_u \times n_u}$ consider the algebraic equation. The algebraic equation is well-posed if it admits a unique solution for all $\mu \in \mathbb{R}^{n_u}$ and if $\mu \mapsto u(\mu)$ is Lipschitz continuous.

Well-posedness of algebraic loops

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Lemma (A sufficient condition)

Consider the algebraic loop for $L \in \mathbb{R}^{n_u \times n_u}$ and $u, \mu \in \mathbb{R}^{n_u}$. If there exists a positive definite matrix $W \in \mathcal{S}^{n_u}_{>0}$ satisfying the matrix inequality

$$\frac{1}{\|W\|}\left(L^T W + WL - 2W\right) < 0,$$

then the algebraic loop is well-posed.

Note that

- the factor $\frac{1}{\|W\|}$ is not necessary, but it indicates how far the inequality is from being violated ($\rightsquigarrow$ robustness margin)

Section 1

$\mathcal{L}_2$-Gain Optimization for Linear Systems

$\mathcal{L}_2$-Gain Optimization for Linear Systems

Recall: $\dot{x} = Ax$

- The origin is exponentially stable
- For $Q > 0$ there exists $P > 0$ satisfying

$$A^T P + PA = -Q$$

- Idea of the Lyapunov equation is that $V(x) = x^T Px$ is a Lyapunov function

$$\dot{V}(x) = x^T (A^T P + PA)x = -x^T Qx < 0, \qquad x \neq 0.$$

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We note that:

- The inequality of the decrease is important not the equality of the Lyapunov equation

⇝ For given A, consider the LMI

$$0 < P$$
$$A^T P + PA < 0$$

instead of the Lyapunov equation

- Advantage: Q is a degree of freedom
- “Optimal” Q and P can be obtained

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LMI (as convex optimization problem):

$$\begin{aligned} \min_{P,\ k} \quad & k \\ \text{subject to} \quad & 0 < k \\ & 0 < P - \alpha I \\ & 0 > P - (k+\alpha) I \\ & 0 > A^T P + PA. \end{aligned}$$

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Here:

- $\alpha > 0$ to ensure that P is not arbitrarily small
- Third constraint to ensure that P is not arbitrarily large

Toolboxes in Matlab:

- CVX, SOSTOOLS, YALMIP

Approximation: $(\varepsilon > 0)$

$$\begin{aligned} \min_{P,\ k} \quad & k \\ \text{subject to} \quad & 0 \leq k \\ & 0 \leq P - \alpha I - \varepsilon I \\ & 0 \geq P - (k+\alpha)I + \varepsilon I \\ & 0 \geq A^T P + PA + \varepsilon I \end{aligned}$$

Asymptotic Stability and $\mathcal{L}_2$-Gain Optimization

Consider:

$$\begin{aligned} \dot{x} &= Ax + Ew \\ z &= Cx + Fw. \end{aligned}$$

Asymptotic Stability and $\mathcal{L}_2$-Gain Optimization

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0-GAS: (internal stability)

- If 0 is globally asymptotically stable for $w = 0$, then the system is called 0-GAS

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Recall: For A Hurwitz, $Q = -2I$, $P > 0$ solution of the Lyap. equation, $V(x) = x^T Px$, we have derived

$$\dot{V}(x) \leq -x^T x + \gamma^2 w^T w, \qquad \gamma = \|PE\|$$

Rearranging terms and integrating (with $x(0) = 0$) yields

$$\begin{aligned}\|x\|^2_{\mathcal{L}_2[0,t)} &\leq \int_0^t x(\tau)^T x(\tau) d\tau + V(x(t)) \\ &\leq \gamma^2 \int_0^t w(\tau)^T w(\tau) d\tau = \gamma^2 \|w\|^2_{\mathcal{L}_2[0,t)}.\end{aligned}$$

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Slight modification: Suppose we can find $P > 0$, so that

$$\dot{V}(x) = x^T (A^T P + PA)x + 2x^T PEw$$
$$< -\gamma \left(\tfrac{1}{\gamma^2} z^T z - w^T w \right), \quad \forall\, (x, w) \neq 0$$

Then we can show that this guarantees

- 0-GAS (since $\dot{V}(x) < 0 \ \ \forall x \neq 0$)
- an $\mathcal{L}_2$-gain bound of $\gamma > 0$ from w to output z; i.e.,

$$\|z\|_{\mathcal{L}_2[0,t)} \leq \gamma \|w\|_{\mathcal{L}_2[0,t)}$$

The bound again follows by integrating (and $x(0) = 0$):

$$\frac{1}{\gamma} \int_0^t z^T(\tau) z(\tau) d\tau + V(x(t)) \leq \gamma \int_0^t w^T(\tau) w(\tau) d\tau$$

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$\rightsquigarrow$ Can we compute $P > 0$ and $\gamma > 0$ by solving an LMI?

Starting point:

$$x^T(A^T P + PA)x + 2x^T PEw + \tfrac{1}{\gamma} z^T z - \gamma w^T w < 0$$

Asymptotic Stability and $\mathcal{L}_2$-Gain Optimization

Starting point:

$$x^T(A^TP + PA)x + 2x^TPEw + \tfrac{1}{\gamma}z^Tz - \gamma w^Tw < 0$$

$$x^T(A^TP + PA + \tfrac{1}{\gamma}C^TC)x + 2x^T(PE + \tfrac{1}{\gamma}C^TF)w + \gamma w^TF^TFw < 0$$

$$\begin{bmatrix} x \\ w \end{bmatrix}^T \left(\begin{bmatrix} A^TP + PA & PE \\ E^TP & -\gamma I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} C^T \\ F^T \end{bmatrix} \begin{bmatrix} C & F \end{bmatrix} \right) \begin{bmatrix} x \\ w \end{bmatrix} < 0$$

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In terms of definite matrices ($0<P$ and):

$$\begin{bmatrix} A^TP+PA & PE\\ E^TP & -\gamma I\end{bmatrix}+\frac{1}{\gamma}\begin{bmatrix} C^T\\ F^T\end{bmatrix}\begin{bmatrix} C & F\end{bmatrix}<0 \qquad (1)$$

Note that:

- For $\gamma>0$, fixed we know how to solve the LMI to obtain P
- However, we would like to minimize $\gamma>0$
- The inequality is not linear in γ

Asymptotic Stability and $\mathcal{L}_2$-Gain Optimization

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Lemma (Schur Complement)

Let $Q\in\mathcal{S}^r$ and $R\in\mathcal{S}^q$ for $r,q\in\mathbb{N}$ and let $S\in\mathbb{R}^{r\times q}$. Then the matrix condition

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}<0$$

is equivalent to the matrix conditions

$$R<0$$

$$Q-SR^{-1}S^T<0.$$

Asymptotic Stability and $\mathcal{L}_2$-Gain Optimization

Starting point:

$$x^T(A^TP + PA)x + 2x^TPEw + \tfrac{1}{\gamma}z^Tz - \gamma w^Tw < 0$$

$$x^T(A^TP + PA + \tfrac{1}{\gamma}C^TC)x + 2x^T(PE + \tfrac{1}{\gamma}C^TF)w + \gamma w^TF^TFw < 0$$

$$\begin{bmatrix} x \\ w \end{bmatrix}^T \left(\begin{bmatrix} A^TP + PA & PE \\ E^TP & -\gamma I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} C^T \\ F^T \end{bmatrix} \begin{bmatrix} C & F \end{bmatrix} \right) \begin{bmatrix} x \\ w \end{bmatrix} < 0$$

In terms of definite matrices ($0 < P$ and):

$$\begin{bmatrix} A^TP + PA & PE \\ E^TP & -\gamma I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} C^T \\ F^T \end{bmatrix} \begin{bmatrix} C & F \end{bmatrix} < 0 \qquad (1)$$

Here, take $R = -\gamma$, $S = \begin{bmatrix} C & F \end{bmatrix}$ and Q as the leftmost matrix.

Then, (1) is equivalent to

$$\left[\begin{array}{cc|c} A^TP + PA & PE & C^T \\ E^TP & -\gamma I & F^T \\ \hline C & F & -\gamma I \end{array}\right] < 0$$

Note that:

- For $\gamma > 0$, fixed we know how to solve the LMI to obtain P
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$$R < 0$$

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Asymptotic Stability and $\mathcal{L}_2$-Gain Optimization (2)

Overall optimization problem:

$$\min_{P,\,\gamma} \quad \gamma$$

subject to

$$0 < P \qquad \text{symmetric}$$

$$0 < \gamma$$

$$0 > \left[\begin{array}{cc|c} A^T P + PA & PE & C^T \\ E^T P & -\gamma I & F^T \\ \hline C & F & -\gamma I \end{array}\right].$$

Notation:

$$\mathrm{He}\; X = X + X^T$$

Reducing redundancy:

$$\mathrm{He}\left[\begin{array}{cc|c} PA & PE & 0 \\ 0 & -\frac{\gamma}{2} I & 0 \\ \hline C & F & -\frac{\gamma}{2} I \end{array}\right] < 0$$

Note that:

- The information that P is symmetric is redundant
- The constraint $0 < \gamma$ is redundant ($R = -\gamma$)
- Don't forget the factor $\frac{1}{2}$ on the diagonal

Lemma (Schur Complement)

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is equivalent to the matrix conditions

$$R < 0$$

$$Q - SR^{-1}S^T < 0.$$

Asymptotic Stability and $\mathcal{L}_2$-Gain Optimization (Example)

Consider:

$$\dot{x} = \begin{bmatrix} -1 & -2 & 2 \\ 1 & -2 & 1 \\ 3 & -2 & -2 \end{bmatrix} x + \begin{bmatrix} 2 & -2 \\ 1 & 3 \\ 3 & -2 \end{bmatrix} w$$

$$z = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} -3 & 2 \end{bmatrix} w.$$

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$$z = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} -3 & 2 \end{bmatrix} w.$$

Solution of the Lyapunov equation with $Q = -2I$:

$$P = \begin{bmatrix} 3.10 & -3.20 & 1.77 \\ -3.20 & 5.50 & -1.80 \\ 1.77 & -1.80 & 1.37 \end{bmatrix}.$$

Estimation (using Young's inequality):

$$\begin{aligned}
\|z\|^2_{\mathcal{L}_2[0,t)} &= \|Cx + Fw\|^2_{\mathcal{L}_2[0,t)} \\
&= \left(\int_0^t x^T CCx + 2x^T C^T Fw + w^T F^T Fw \, d\tau \right)^2 \\
&\leq \left(\int_0^t 2x^T CCx + 2w^T F^T Fw \, d\tau \right)^2 \\
&\leq \left(\int_0^t 2\lambda_{\max}(C^T C) x^T x + 2\lambda_{\max}(F^T F) w^T w \, d\tau \right)^2 \\
&= 2\lambda_{\max}(C^T C)\|x\|^2_{\mathcal{L}_2[0,t)} + 2\lambda_{\max}(F^T F)\|w\|^2_{\mathcal{L}_2[0,t)} \\
&\leq 2 \left(\lambda_{\max}(C^T C)\|PE\|^2 + \lambda_{\max}(F^T F) \right) \|w\|^2_{\mathcal{L}_2[0,t)} \\
&= 2711 \cdot \|w\|^2_{\mathcal{L}_2[0,t)}.
\end{aligned}$$

Hence, $\gamma = \sqrt{2711} = 52.07$

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$$z = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} -3 & 2 \end{bmatrix} w.$$

Alternatively,

$$\gamma = 7.43$$

$$P = \begin{bmatrix} 4.38 & -0.22 & -4.12 \\ -0.22 & 0.32 & -0.02 \\ -4.12 & -0.02 & 4.18 \end{bmatrix}.$$

is returned as the solution of the LMI

Solution of the Lyapunov equation with $Q = -2I$:

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Estimation (using Young's inequality):

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&\leq \left(\int_0^t 2x^T CCx + 2w^T F^T Fw \, d\tau \right)^2 \\
&\leq \left(\int_0^t 2\lambda_{\max}(C^T C) x^T x + 2\lambda_{\max}(F^T F) w^T w \, d\tau \right)^2 \\
&= 2\lambda_{\max}(C^T C) \|x\|^2_{\mathcal{L}_2[0,t)} + 2\lambda_{\max}(F^T F) \|w\|^2_{\mathcal{L}_2[0,t)} \\
&\leq 2 \left(\lambda_{\max}(C^T C) \|PE\|^2 + \lambda_{\max}(F^T F) \right) \|w\|^2_{\mathcal{L}_2[0,t)} \\
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\end{aligned}$$

Hence, $\gamma = \sqrt{2711} = 52.07$

Feedback Synthesis

Consider: ($\rightsquigarrow$ Design static state feedback K)

$$\begin{aligned} \dot{x} &= Ax + Bu + Ew \\ z &= Cx + Du + Fw \\ u &= Kx. \end{aligned}$$

Feedback Synthesis

Consider: ($\rightsquigarrow$ Design static state feedback K)

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In closed-loop form:

$$\begin{aligned}\dot{x} &= (A + BK)x + Ew\\ z &= (C + DK)x + Fw.\end{aligned}$$

Same approach as before: ($P > 0$, $V(x) = x^T Px$)

$$\begin{aligned}\dot{V}(x) = x^T &\left((A + BK)^T P + P(A + BK)\right) x + 2x^T PEw\\ &< -\gamma\left(\tfrac{1}{\gamma^2} z^T z - w^T w\right) \quad \forall\, (x, w) \neq 0\end{aligned}$$

(Unknowns: P, K, γ)

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Define $\Lambda = P^{-1}$: (left and right multiplication with Λ)

$$\begin{aligned}\Lambda A^T P\Lambda + \Lambda PA\Lambda + \Lambda K^T B^T P\Lambda + \Lambda PBK\Lambda\\ = \Lambda A^T + A\Lambda + \Lambda(BK)^T + BK\Lambda\\ = \operatorname{He}(A\Lambda + BK\Lambda)\end{aligned}$$

Define $X = K\Lambda$. Then

$$\operatorname{He}(A\Lambda + BK\Lambda) = \operatorname{He}(A\Lambda + BX)$$

is linear in the unknowns $\Lambda = P^{-1}$ and $X = K\Lambda$

The condition

$$\operatorname{He}(A\Lambda + BX) < 0$$

guarantees that $A + BK$ is Hurwitz (with Lyapunov function $V(x) = x^T P x$)

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STEP 2: $\mathcal{L}_2$-gain optimization

Recall the condition (for known K)

$$\mathrm{He}\left[\begin{array}{cc|c} P(A+BK) & PE & 0 \\ 0 & -\frac{\gamma}{2}I & 0 \\ \hline C+DK & F & -\frac{\gamma}{2}I \end{array}\right] < 0$$

Feedback Synthesis (2)

Recall the condition:

$$\mathrm{He}\left[\begin{array}{cc|c} P(A+BK) & PE & 0 \\ 0 & -\frac{\gamma}{2}I & 0 \\ \hline C+DK & F & -\frac{\gamma}{2}I \end{array}\right] < 0$$

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Define $\Lambda = P^{-1}$ (left and right multiplication):

$$\begin{bmatrix} \Lambda & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \operatorname{He}\left[\begin{array}{cc|c} P(A+BK) & PE & 0 \\ 0 & -\frac{\gamma}{2}I & 0 \\ \hline C+DK & F & -\frac{\gamma}{2}I \end{array}\right] \begin{bmatrix} \Lambda & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} = \operatorname{He}\left[\begin{array}{cc|c} A\Lambda+BK\Lambda & E & 0 \\ 0 & -\frac{\gamma}{2}I & 0 \\ \hline C\Lambda+DK\Lambda & F & -\frac{\gamma}{2}I \end{array}\right] < 0$$

Feedback Synthesis (2)

Recall the condition:

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Again, define $X = K\Lambda$:

$$\operatorname{He}\left[\begin{array}{cc|c} A\Lambda+BX & E & 0 \\ 0 & -\frac{\gamma}{2}I & 0 \\ \hline C\Lambda+DX & F & -\frac{\gamma}{2}I \end{array}\right] < 0.$$

Feedback Synthesis (2)

Recall the condition:

$$\operatorname{He}\left[\begin{array}{cc|c} P(A+BK) & PE & 0 \\ 0 & -\frac{\gamma}{2}I & 0 \\ \hline C+DK & F & -\frac{\gamma}{2}I \end{array}\right] < 0$$

Define $\Lambda = P^{-1}$ (left and right multiplication):

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Corresponding optimization/feasibility problem:

$$\begin{aligned} \min_{\Lambda, X, \gamma} \quad & \gamma \\ \text{subject to} \quad & 0 < \Lambda \qquad \text{symmetric} \\ & 0 < \gamma \\ & 0 > \operatorname{He}\left[\begin{array}{cc|c} (A\Lambda+BX) & E & 0 \\ 0 & -\frac{\gamma}{2}I & 0 \\ \hline C\Lambda+DX & F & -\frac{\gamma}{2}I \end{array}\right] \end{aligned}$$

$\rightsquigarrow$ Lyapunov function $V(x) = x^T\Lambda^{-1}x$ and a feedback gain matrix $K = X\Lambda^{-1}$ such that γ is minimal

Feedback Synthesis (Example)

Consider:

$$\dot{x} = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 2 & -1 \\ -3 & 2 & 2 \end{bmatrix} x + \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} u + \begin{bmatrix} 2 & -2 \\ 1 & 3 \\ 3 & -2 \end{bmatrix} w$$

$$z = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} -3 & 2 \end{bmatrix} w.$$

Solution of the LMI:

$$\gamma = 8.1910$$

$$K = \begin{bmatrix} -7.32 & 6.64 & 4.62 \end{bmatrix}$$

$$P = \begin{bmatrix} 1.18 & -1.19 & -0.62 \\ -1.19 & 1.33 & 0.71 \\ -0.62 & 0.71 & 0.40 \end{bmatrix}.$$

Eigenvalues of A:

$$\{4, 0.5 \pm 1.32j\}$$

Eigenvalues of $A + BK$:

$$\{-4.78 \pm 0.90j, -0.15\}$$

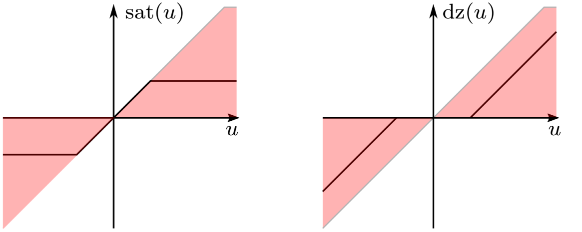
Eigenvalues of P:

$$\{0.02, 0.07, 2.34\}$$

Section 2

Systems with Saturation

Systems with Saturation



Consider:

$$\dot{x} = Ax + B\,\mathrm{sat}(u)$$
$$u = Kx$$

Saturation: (we will suppress the limit $\bar{u}$ in the following)

$$\mathrm{sat}(u) \doteq \begin{cases} -1, & u < -1 \\ u, & -1 \leq u \leq 1 \\ 1, & 1 < u. \end{cases}$$

$$\mathrm{sat}_{\bar{u}}(u) \doteq \begin{cases} -\bar{u}, & u < -\bar{u} \\ u, & -\bar{u} \leq u \leq \bar{u} \\ \bar{u}, & \bar{u} < u. \end{cases}$$

Deadzone: ($q = \mathrm{dz}(u)$)

$\mathrm{dz}(u) = u - \mathrm{sat}(u)$ and $\mathrm{dz}_{\bar{u}}(u) = u - \mathrm{sat}_{\bar{u}}(u)$,

We assume to have decentralized saturations

- i.e., for $u \in \mathbb{R}^{n_u}$ we assume that each input has its own saturation function, possibly with different saturation levels $\bar{u}_i$ on the i^{th} input.

Systems with Saturation

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Deadzone: $(q = \mathrm{dz}(u))$

$$\mathrm{dz}(u) = u - \mathrm{sat}(u) \qquad \text{and} \qquad \mathrm{dz}_{\bar{u}}(u) = u - \mathrm{sat}_{\bar{u}}(u),$$

We assume to have decentralized saturations

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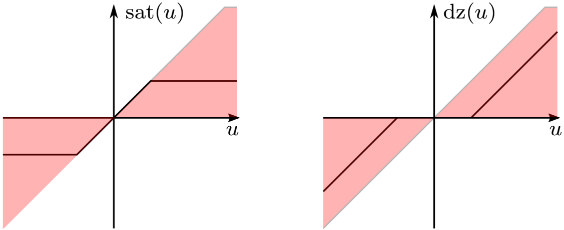

Note that:

- $u \in \mathbb{R}$ and $q = u - \mathrm{sat}(u)$, satisfies
$$\mathrm{dz}(u)\,\mathrm{sat}(u) \geq 0 \qquad \text{or equivalently} \qquad q(u-q) \geq 0.$$
- In particular $\mathrm{sign}(\mathrm{dz}(u)) = \mathrm{sign}(\mathrm{sat}(u))$ or equivalently $\mathrm{sign}(q) = \mathrm{sign}(u-q)$
- Moreover,
$$wq(u-q) \geq 0 \qquad \text{for } w > 0$$
- A vector version: ($W > 0$, diagonal)
$$\mathrm{dz}(u)^T W\,\mathrm{sat}(u) \geq 0, \qquad q^T W(u-q) \geq 0$$

LMI-Based Saturated Linear State Feedback Design

Controller design $u = Kx$: $(q = u - \mathrm{sat}(u))$

$$\dot{x} = Ax + B\,\mathrm{sat}(Kx) = (A + BK)x - B\,\mathrm{dz}(Kx)$$

Consider Lyapunov function: $(P > 0)$

$$V(x) = x^T P x$$

We want: $\dot{V}(x(t)) < 0$ despite the nonlinearity.

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Formally, for $(x, q) = (x, \text{dz}(Kx)) \neq 0$ we want

$$\begin{bmatrix} x \\ q \end{bmatrix}^T \text{He} \begin{bmatrix} 0 & 0 \\ -WK & W \end{bmatrix} \begin{bmatrix} x \\ q \end{bmatrix} = 2q^T W(u - q)$$

$$= 2(\text{dz}(Kx))^T W(Kx - \text{dz}(Kx)) \geq 0 \quad \Rightarrow \quad \dot{V}(x) < 0$$

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Derivative of the candidate Lyapunov function

$$\dot{V}(x) = x^T((A + BK)^T P + P(A + BK))x - 2x^T PBq$$

$$= x^T(A^T P + PA + K^T B^T P + PBK)x - 2x^T PB\,\mathrm{dz}(Kx)$$

$$= \begin{bmatrix} x \\ q \end{bmatrix}^T \mathrm{He} \begin{bmatrix} PA + PBK & -PB \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ q \end{bmatrix}$$

Unknowns: $W > 0$ diagonal; $P > 0$; K

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Unknowns: $W > 0$ diagonal; $P > 0$; K

Lemma (S-Lemma or S-Procedure)

Let $M_0, M_1 \in \mathcal{S}^r$, $r \in \mathbb{N}$, and suppose there exists $\zeta^* \in \mathbb{R}^r$ such that $(\zeta^*)^T M_1 \zeta^* > 0$. Then the following statements are equivalent:

1. *There exists $\tau > 0$ such that $M_0 - \tau M_1 > 0$.*
2. *For all $\zeta \neq 0$ such that $\zeta^T M_1 \zeta \geq 0$ it holds that $\zeta^T M_0 \zeta > 0$.*

LMI-Based Saturated Linear State Feedback Design

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$$x^T(A^T P + PA + K^T B^T P + PBK)x - 2x^T PBq$$
$$+2q^T W(Kx - q) < 0$$

$\rightsquigarrow$ The inequality guarantees what we want ("1. $\Rightarrow$ 2.")

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$$x^T(A^T P + PA + K^T B^T P + PBK)x - 2x^T PBq + 2q^T W(Kx - q) < 0$$

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In Matrix notation:

$$\begin{bmatrix} x \\ q \end{bmatrix}^T \mathrm{He} \begin{bmatrix} PA + PBK & -PB \\ WK & -W \end{bmatrix} \begin{bmatrix} x \\ q \end{bmatrix} < 0$$

is negative definite.

LMI-Based Saturated Linear State Feedback Design (2)

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If there exists K, P, W such that

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then $u = Kx$ defines a stabilizing control law.

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then $u = Kx$ defines a stabilizing control law.
Multiplying with $\mathrm{diag}(P^{-1}, W^{-1})$:

$$0 > \begin{bmatrix} P^{-1} & 0 \\ 0 & W^{-1} \end{bmatrix} \mathrm{He}\begin{bmatrix} PA + PBK & -PB \\ WK & -W \end{bmatrix}\begin{bmatrix} P^{-1} & 0 \\ 0 & W^{-1} \end{bmatrix}$$

$$= \mathrm{He}\begin{bmatrix} AP^{-1} + BKP^{-1} & -BW^{-1} \\ KP^{-1} & W^{-1} \end{bmatrix} = \mathrm{He}\begin{bmatrix} A\Lambda + BX & -BD \\ X & D \end{bmatrix}$$

where $\Lambda = P^{-1}$, $X = KP^{-1}$, $D = W^{-1}$

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where $\Lambda = P^{-1}$, $X = KP^{-1}$, $D = W^{-1}$
If there exist Λ, X, D so that

$$\operatorname{He}\begin{bmatrix} A\Lambda + BX & -BD \\ X & -D \end{bmatrix} < 0, \quad 0 < \Lambda, \quad 0 < D \text{ diagonal}$$

then the bounded input $\operatorname{sat}(Kx)$, $K = X\Lambda^{-1}$ globally asymptotically stabilizes the origin.

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$$\dot{x} = Ax + B\,\mathrm{sat}(Kx) = (A + BK)x - B\,\mathrm{dz}(Kx)$$

If there exists K, P, W such that

$$\begin{bmatrix} x \\ q \end{bmatrix}^T \mathrm{He} \begin{bmatrix} PA + PBK & -PB \\ WK & -W \end{bmatrix} \begin{bmatrix} x \\ q \end{bmatrix} < 0$$

then $u = Kx$ defines a stabilizing control law.
Multiplying with $\mathrm{diag}(P^{-1}, W^{-1})$:

$$0 > \begin{bmatrix} P^{-1} & 0 \\ 0 & W^{-1} \end{bmatrix} \mathrm{He} \begin{bmatrix} PA + PBK & -PB \\ WK & -W \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ 0 & W^{-1} \end{bmatrix}$$

$$= \mathrm{He} \begin{bmatrix} AP^{-1} + BKP^{-1} & -BW^{-1} \\ KP^{-1} & W^{-1} \end{bmatrix} = \mathrm{He} \begin{bmatrix} A\Lambda + BX & -BD \\ X & D \end{bmatrix}$$

where $\Lambda = P^{-1}$, $X = KP^{-1}$, $D = W^{-1}$
If there exist Λ, X, D so that

$$\mathrm{He} \begin{bmatrix} A\Lambda + BX & -BD \\ X & -D \end{bmatrix} < 0, \quad 0 < \Lambda, \quad 0 < D \text{ diagonal}$$

then the bounded input $\mathrm{sat}(Kx)$, $K = X\Lambda^{-1}$ globally asymptotically stabilizes the origin.

How likely is it that the LMI has a solution?

LMI-Based Saturated Linear State Feedback Design (2)

Controller design $u = Kx$: ($q = u - \mathrm{sat}(u)$)

$$\dot{x} = Ax + B\,\mathrm{sat}(Kx) = (A + BK)x - B\,\mathrm{dz}(Kx)$$

If there exists K, P, W such that

$$\begin{bmatrix} x \\ q \end{bmatrix}^T \mathrm{He} \begin{bmatrix} PA + PBK & -PB \\ WK & -W \end{bmatrix} \begin{bmatrix} x \\ q \end{bmatrix} < 0$$

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then the bounded input $\mathrm{sat}(Kx)$, $K = X\Lambda^{-1}$ globally asymptotically stabilizes the origin.

How likely is it that the LMI has a solution?

- If the LMI has a solution then

$$0 > \begin{bmatrix} I & -B \\ 0 & I \end{bmatrix} \mathrm{He} \begin{bmatrix} A\Lambda + BX & -BD \\ X & -D \end{bmatrix} \begin{bmatrix} I & 0 \\ -B^T & I \end{bmatrix} = \mathrm{He} \begin{bmatrix} A\Lambda & BD \\ X & -D \end{bmatrix}.$$

- Then the Schur complement implies that

$$A\Lambda + \Lambda A^T = -Q < 0$$

$$2D + (X^T + BD)Q^{-1}(X + D^T B^T) < 0$$

⇝ A is Hurwitz (since $\Lambda > 0$)

⇝ The origin is globally asymptotically stable with $K = 0$.

⇝ We need local approaches

Global Asymptotic Stability Analysis

Consider

$$\begin{aligned} \dot{x} &= Ax + Bq + Ew \\ z &= Cx + Dq + Fw \\ u &= Kx + Lq + Gw \\ q &= u - \mathrm{sat}(u) \end{aligned}$$

but we start with $w = 0$.

For K, L given, how do we establish global asymptotic stability of the orign?

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For K, L given, how do we establish global asymptotic stability of the orign?

- Candidate Lyapunov function $V(x) = x^T Px$
- We want that the sector condition implies a decrease, i.e.,

$$q^T W(u - q) \geq 0 \quad \Rightarrow$$

$$\dot{V}(x) = x^T(A^T P + PA)x + 2x^T PBq < 0, \ (x, q) \neq 0$$

- S-Procedure: ($W > 0$ diagonal)

$$x^T(A^T P + PA)x + 2x^T PBq + 2q^T W(u - q) < 0$$

- Using the definition of u:

$$x^T(A^T P + PA)x + 2x^T PBq \\ + 2q^T W(Kx - Lq - q) < 0$$

- Corresponding LMI:

$$\mathrm{He}\begin{bmatrix} PA & PB \\ WK & -W + WL \end{bmatrix} < 0.$$

⇝ Feasibility (unknowns $P > 0$, $W > 0$ diagonal) implies global asymptotic stability

Global Asymptotic Stability Analysis

Consider

$$\begin{aligned}\dot{x} &= Ax + Bq + Ew\\ z &= Cx + Dq + Fw\\ u &= Kx + Lq + Gw\\ q &= u - \operatorname{sat}(u)\end{aligned}$$

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$$\operatorname{He}\begin{bmatrix} PA & PB \\ WK & -W + WL \end{bmatrix} < 0.$$

$\rightsquigarrow$ Feasibility (unknowns $P > 0$, $W > 0$ diagonal) implies global asymptotic stability

- Note that: The Schur complement implies

$$-2W + WL + L^T W < 0$$

Recall: Well-posedness of the algebraic loop

$$u = Kx + L(u - \operatorname{sat}(u)) + Gw$$

Lemma (A sufficient condition)

Consider the algebraic loop for $L \in \mathbb{R}^{n_u \times n_u}$ and $u, \mu \in \mathbb{R}^{n_u}$. If there exists a positive definite matrix $W \in \mathcal{S}^{n_u}_{>0}$ satisfying the matrix inequality

$$\frac{1}{\|W\|}\left(L^T W + WL - 2W\right) < 0,$$

then the algebraic loop is well-posed.

Global Asymptotic Stability Analysis (2)

Consider (with $w = 0$)

$$\begin{aligned} \dot{x} &= Ax + Bq + Ew \\ z &= Cx + Dq + Fw \\ u &= Kx + Lq + Gw \\ q &= u - \mathrm{sat}(u) \end{aligned}$$

- Corresponding LMI:

$$\mathrm{He}\begin{bmatrix} PA & PB \\ WK & -W + WL \end{bmatrix} < 0.$$

- Note that: The Schur complement implies

$$-2W + WL + L^T W < 0$$

Recall: Well-posedness of the algebraic loop

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$$\frac{1}{\|W\|}\left(L^T W + WL - 2W\right) < 0,$$

then the algebraic loop is well-posed.

Introducing a robustness margin: ($\nu \in (0, 1]$)

$$x^T(A^T P + PA)x + 2x^T PBq + 2q^T W(Kx - Lq - q) \leq$$

$$x^T(A^T P + PA)x + 2x^T PBq + 2q^T W(Kx - Lq - \nu q) \leq 0$$

Corresponding LMI:

$$\mathrm{He}\begin{bmatrix} PA & PB \\ WK & -\nu W + WL \end{bmatrix} < 0 \qquad (2)$$

The Schur Complement implies:

$$-2\nu W + WL + L^T W < 0.$$

If (2) is satisfied, then

$$\frac{1}{\|W\|}\left(L^T W + WL - 2W\right) < -2(1-\nu)\frac{W}{\|W\|} < 0$$

$\mathcal{L}_2$-Stability and $\mathcal{L}_2$-Gain Optimization

Consider

$$\begin{array}{rcl} \dot{x} & = & Ax + Bq + Ew \\ z & = & Cx + Dq + Fw \\ u & = & Kx + Lq + Gw \\ q & = & u - \mathrm{sat}(u) \end{array}$$

$\mathcal{L}_2$-stability problem (for given K, L). We perform the same steps as before

$$q^T W(u-q) \geq 0 \qquad \Rightarrow \qquad x^T(A^TP + PA)x + 2x^TP(Bq + Ew) < -\gamma\left(\frac{1}{\gamma^2}z^Tz - w^Tw\right)$$

(S-Procedure) $\quad x^T(A^TP + PA)x + 2x^TP(Bq + Ew) + 2q^TW(u-q) + \gamma\left(\frac{1}{\gamma^2}z^Tz - w^Tw\right) < 0$

(Definition of u) $\quad x^T(A^TP + PA)x + 2x^TP(Bq + Ew) + 2q^TW(Gw + Kx + Lq - q) + \frac{1}{\gamma}z^Tz - \gamma w^Tw < 0$

Corresponding matrix notation:

$$\begin{bmatrix} x \\ q \\ w \end{bmatrix}^T \left(\begin{bmatrix} A^TP + PA & PB + K^TW & PE \\ B^TP + WK & -2W + WL + L^TW & WG \\ E^TP & G^TW & -\gamma I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} C^T \\ D^T \\ F^T \end{bmatrix} [C \;\; D \;\; F] \right) \begin{bmatrix} x \\ q \\ w \end{bmatrix} < 0.$$

Schur complement

$$\mathrm{He}\left[\begin{array}{ccc|c} PA & PB & PE & 0 \\ WK & -W + WL & WG & 0 \\ 0 & 0 & -\frac{\gamma}{2}I & 0 \\ \hline C & D & F & -\frac{\gamma}{2}I \end{array}\right] < 0.$$

$\mathcal{L}_2$-Stability and $\mathcal{L}_2$-Gain Optimization (Optimization Problem & Example)

Overall Optimization problem: $(\nu \in (0,1])$

$$
\begin{aligned}
\min_{P,W,\gamma} \quad & \gamma \\
\text{subject to} \quad & 0 < P \quad \text{symmetric} \\
& 0 < W \quad \text{diagonal} \\
& 0 < \gamma \\
& 0 > \mathrm{He} \left[\begin{array}{ccc|c} PA & PB & PE & 0 \\ WK & -\nu W + WL & WG & 0 \\ 0 & 0 & -\frac{\gamma}{2} I & 0 \\ \hline C & D & F & -\frac{\gamma}{2} I \end{array} \right]
\end{aligned}
$$

Example:
Consider

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} -1 & -2 & 2 \\ 1 & -2 & 1 \\ 3 & -2 & -2 \end{bmatrix} x + \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} q + \begin{bmatrix} 2 & -2 \\ 1 & 3 \\ 3 & -2 \end{bmatrix} w \\
z &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} -3 & 2 \end{bmatrix} w \\
u &= \begin{bmatrix} -1 & -2 & 1 \end{bmatrix} x + \begin{bmatrix} 2 & -3 \end{bmatrix} w \\
q &= u - \mathrm{sat}(u).
\end{aligned}
$$

Solution: $\gamma = 7.8607$ (for $\nu = 1$)

Section 3

Regional Analysis

Regional Analysis

Lemma

Let $u \in \mathbb{R}$ and $q = u - \mathrm{sat}(u)$. For an arbitrary row vector $H \in \mathbb{R}^{1\times n}$, for all $x \in \mathbb{R}^n$ such that $\mathrm{sat}(Hx) = Hx$, the sector condition

$$(u - q + Hx)q \geq 0 \qquad \textit{holds.}$$

Proof.

Let $x \in \mathbb{R}^n$ such that $\mathrm{sat}(Hx) = Hx$ is satisfied. Then

$$\begin{aligned}(u - q + Hx)q &= (u - u + \mathrm{sat}(u) + Hx)(u - \mathrm{sat}(u)) \\ &= (\mathrm{sat}(u) + Hx)(u - \mathrm{sat}(u)) \\ &= (\mathrm{sat}(u) + \mathrm{sat}(Hx))(u - \mathrm{sat}(u))\end{aligned}$$

Now, consider two cases:

1. $u = \mathrm{sat}(u)$: Then $(\mathrm{sat}(u) + \mathrm{sat}(Hx))(u - \mathrm{sat}(u)) = 0$
2. $u \neq \mathrm{sat}(u)$: Both terms on the right have the same sign or are zero. Specifically: (1) if $u > \mathrm{sat}(u)$ then $\mathrm{sat}(u) \geq \mathrm{sat}(Hx)$; (2) if $u = \mathrm{sat}(u)$ the right term vanishes; (3) if $u < \mathrm{sat}(u)$ then $\mathrm{sat}(u) \leq \mathrm{sat}(Hx)$.

□

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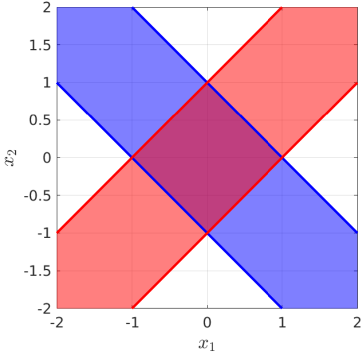
□

Vector version and positive scaling $W > 0$ diagonal:

$$(u - q + Hx)^T W q \geq 0$$

Visualization of the sector condition:



Domain where $\mathrm{sat}(Hx) = Hx$ is satisfied. Here $H_b = [1, 1]$ (blue) and $H_r = [1, -1]$ (red).

Local Asymptotic Stability

Candidate Lyapunov function:

$$V(x) = x^T P x$$

We want

$$V(x) = x^T P x \leq 1 \quad \Rightarrow \quad \dot{V}(x) < 0, \quad x \neq 0$$

Note that: ($\bar{u}_i$ saturation level)

$$\tfrac{1}{\bar{u}_i^2} x^T H_i^T H_i x < x^T P x, \qquad \forall\, x \neq 0, \quad \forall i = 1, \ldots, n_u$$

implies ($i \in \{1, \ldots, n_u\}$)

$$\mathrm{sat}_{\bar{u}_i}(H_i x) = H_i x \quad \forall\, x \in \{x \in \mathbb{R}^{n_u} |\, x^T P x \leq 1\}$$

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$$\mathrm{sat}_{\bar{u}_i}(H_i x) = H_i x \quad \forall\, x \in \{x \in \mathbb{R}^{n_u} \,|\, x^T P x \leq 1\}$$

Schur complement: (unknowns P, H_i)

$$0 < \begin{bmatrix} P & H_i^T \\ H_i & \bar{u}_i^2 \end{bmatrix}, \quad i = 1, \ldots, n_u$$

Consider (with $w = 0$)

$$\begin{aligned} \dot{x} &= Ax + Bq + Ew \\ z &= Cx + Dq + Fw \\ u &= Kx + Lq + Gw \\ q &= u - \mathrm{sat}(u) \end{aligned}$$

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$\rightsquigarrow$ We can proceed as before.

We want that the sector condition implies a decrease, i.e.,

$$q^T W(u - q + Hx) \geq 0 \quad \Rightarrow$$
$$\dot{V}(x) = x^T(A^T P + PA)x + 2x^T PBq < 0, \ (x, q) \neq 0$$

and we apply the S-procedure

$$x^T(A^T P + PA)x + 2x^T PBq$$
$$+ 2q^T W(Kx - Lq - q + Hx) < 0 \quad \forall\, (x, q) \neq 0$$

and obtain the matrix representation

$$\mathrm{He} \begin{bmatrix} PA & PB \\ WK + WH & -W + WL \end{bmatrix} < 0$$

$\rightsquigarrow$ Not an LMI due to WH

Local Asymptotic Stability (2)

Define set of new unknowns:

$$\Lambda_1 = P^{-1}, \qquad \Lambda_2 = W^{-1}, \qquad \Gamma = HP^{-1}$$

Note that:

$$\begin{bmatrix} \Gamma_1 \\ \vdots \\ \Gamma_{n_u} \end{bmatrix} = \Gamma = HP^{-1} = \begin{bmatrix} H_1 P^{-1} \\ \vdots \\ H_{n_u} P^{-1} \end{bmatrix}$$

$$\begin{aligned} 0 &< \begin{bmatrix} P^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P & H_i^T \\ H_i & \bar{u}_i^2 \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} P^{-1} & P^{-1} H_i^T \\ H_i P^{-1} & \bar{u}_i^2 \end{bmatrix} = \begin{bmatrix} \Lambda_1 & \Gamma_i^T \\ \Gamma_i & \bar{u}_i^2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} 0 &> \begin{bmatrix} P^{-1} & 0 \\ 0 & W^{-1} \end{bmatrix} \mathrm{He} \begin{bmatrix} PA & PB \\ WK + WH & -W + WL \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ 0 & W^{-1} \end{bmatrix} \\ &= \mathrm{He} \begin{bmatrix} AP^{-1} & BW^{-1} \\ KP^{-1} + HP^{-1} & -W^{-1} + LW^{-1} \end{bmatrix} \\ &= \mathrm{He} \begin{bmatrix} A\Lambda_1 & B\Lambda_2 \\ K\Lambda_1 + \Gamma & -\Lambda_2 + L\Lambda_2 \end{bmatrix} \end{aligned}$$

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Note that:

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$$0 < \begin{bmatrix} P^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P & H_i^T \\ H_i & \bar{u}_i^2 \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} P^{-1} & P^{-1}H_i^T \\ H_i P^{-1} & \bar{u}_i^2 \end{bmatrix} = \begin{bmatrix} \Lambda_1 & \Gamma_i^T \\ \Gamma_i & \bar{u}_i^2 \end{bmatrix}$$

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$$= \mathrm{He} \begin{bmatrix} AP^{-1} & BW^{-1} \\ KP^{-1} + HP^{-1} & -W^{-1} + LW^{-1} \end{bmatrix}$$

$$= \mathrm{He} \begin{bmatrix} A\Lambda_1 & B\Lambda_2 \\ K\Lambda_1 + \Gamma & -\Lambda_2 + L\Lambda_2 \end{bmatrix}$$

Corresponding optimization problem:

$$\min_{\Lambda_1, \Lambda_2, \Gamma, k} \quad k$$

subject to
$0 < \Lambda_1$ symmetric
$0 < \Lambda_2$ diagonal
$0 < k$
$0 < kI - \Lambda_1$

$$0 < \mathrm{He} \begin{bmatrix} \frac{1}{2}\Lambda_1 & 0 \\ \Gamma_i & \frac{1}{2}\bar{u}_i^2 \end{bmatrix}, \quad i = 1, \ldots, n_u$$

$$0 > \mathrm{He} \begin{bmatrix} A\Lambda_1 & B\Lambda_2 \\ K\Lambda_1 + \Gamma & -\Lambda_2 + L\Lambda_2 \end{bmatrix}$$

- If the the optimization problem is feasible then the origin is locally asymptotically stable
- Estimate of the region of attraction $\{x \in \mathbb{R}^n : x^T P x \leq 1\}$ (with $P = \Lambda_1^{-1}$)
- The smallest eigenvalue of P is maximized:

$$0 < kI - \Lambda_1 \quad \Longleftrightarrow \quad \tfrac{1}{k} I < P$$

- $\nu \in (0, 1)$ can be incorporated

Local Asymptotic Stability (Example)

Consider

$$\dot{x} = Ax + B\,\mathrm{sat}(u) + Ew$$
$$z = Cx + D\,\mathrm{sat}(u) + Fw$$
$$u = Kx.$$

Using the deadzone operator:

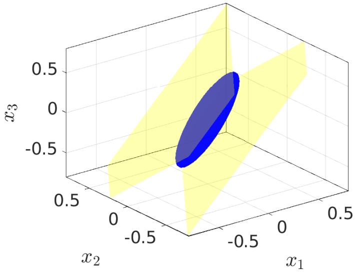
$$\dot{x} = Ax + BKx - Bq + Ew = (A + BK)x - Bq + Ew$$
$$z = Cx + DBx - Dq + Fw = (C + DB)x - Dq + Fw$$
$$u = Kx$$
$$q = u - \mathrm{sat}(u)$$

We continue with an earlier example (which we have stabilized in the unconstrained case):

$$\dot{x} = \begin{bmatrix} -20.93 & 21.92 & 11.83 \\ -15.62 & 15.28 & 8.22 \\ 4.31 & -4.64 & -2.61 \end{bmatrix} x + \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} q + \begin{bmatrix} 2 & -2 \\ 1 & 3 \\ 3 & -2 \end{bmatrix} w$$
$$z = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} -3 & 2 \end{bmatrix} w$$
$$u = \begin{bmatrix} -7.31 & 6.64 & 4.61 \end{bmatrix} x + \begin{bmatrix} 2 & -3 \end{bmatrix} w$$
$$q = u - \mathrm{sat}(u)$$

- (For $w = 0$) the system without saturation is globally asymptotically stable
- Saturation level $\bar{u} = 1$, $(|\,\mathrm{sat}(Hx)| = 1)$
- We obtain $V(x) = x^T P x$ with

$$P = \begin{bmatrix} 21.93 & -22.11 & -10.78 \\ -22.11 & 26.01 & 12.83 \\ -10.78 & 12.83 & 9.43 \end{bmatrix}$$
$$H = [2.75, -1.98, -2.06]$$



$\mathcal{L}_2$-Stability and $\mathcal{L}_2$-Gain Optimization

We continue with:

$$\begin{aligned} \dot{x} &= Ax + Bq + Ew \\ z &= Cx + Dq + Fw \\ u &= Kx + Lq + Gw \\ q &= u - \mathrm{sat}(u) \end{aligned}$$

- Local asymptotic stability: $V(x) = x^T Px$,

$$\dot{V}(x) < 0 \quad \forall x \in \{x \in \mathbb{R}^n \backslash \{0\} |\; V(x) \leq 1\}$$

$\mathcal{L}_2$-Stability and $\mathcal{L}_2$-Gain Optimization

We continue with:

$$\begin{aligned}
\dot{x} &= Ax + Bq + Ew \\
z &= Cx + Dq + Fw \\
u &= Kx + Lq + Gw \\
q &= u - \text{sat}(u)
\end{aligned}$$

- Local asymptotic stability: $V(x) = x^T P x$,

$$\dot{V}(x) < 0 \quad \forall x \in \{x \in \mathbb{R}^n \backslash \{0\} | \ V(x) \leq 1\}$$

- Local $\mathcal{L}_2$-stability: For $s > 0$ fixed, find $V(x) = x^T P x$ and $\gamma > 0$ such that

$$\|z\|_{\mathcal{L}_2} \leq \gamma \|w\|_{\mathcal{L}_2} \qquad \forall \ \|w\|_{\mathcal{L}_2} \leq s$$

(and $x(0) = 0$).

- Derive conditions based on

$$\dot{V}(x(t)) \leq w^T w \qquad \forall x \in \{x \in \mathbb{R}^n : V(x) \leq s^2\}$$

- Corresponding LMI:

$$0 < \begin{bmatrix} P & H_i^T \\ H_i & \frac{\bar{u}_i^2}{s^2} \end{bmatrix}, \quad i = 1, \ldots, n_u.$$

$\mathcal{L}_2$-Stability and $\mathcal{L}_2$-Gain Optimization

We continue with:

$$\begin{aligned} \dot{x} &= Ax + Bq + Ew \\ z &= Cx + Dq + Fw \\ u &= Kx + Lq + Gw \\ q &= u - \text{sat}(u) \end{aligned}$$

- Local asymptotic stability: $V(x) = x^T P x$,

$$\dot{V}(x) < 0 \quad \forall x \in \{x \in \mathbb{R}^n \backslash \{0\} |\, V(x) \leq 1\}$$

- Local $\mathcal{L}_2$-stability: For $s > 0$ fixed, find $V(x) = x^T P x$ and $\gamma > 0$ such that

$$\|z\|_{\mathcal{L}_2} \leq \gamma \|w\|_{\mathcal{L}_2} \qquad \forall\ \|w\|_{\mathcal{L}_2} \leq s$$

(and $x(0) = 0$).

- Derive conditions based on

$$\dot{V}(x(t)) \leq w^T w \qquad \forall x \in \{x \in \mathbb{R}^n : V(x) \leq s^2\}$$

- Corresponding LMI:

$$0 < \begin{bmatrix} P & H_i^T \\ H_i & \frac{\bar{u}_i^2}{s^2} \end{bmatrix}, \quad i = 1, \ldots, n_u.$$

Overall optimization problem:

$$\begin{aligned} &\min_{\Lambda_1, \Lambda_2, \Gamma, \delta} && \delta \\ &\text{subject to} && 0 < \Lambda_1 \quad \text{symmetric} \\ & && 0 < \Lambda_2 \quad \text{diagonal} \\ & && 0 < \delta \\ & && 0 < \text{He} \begin{bmatrix} \frac{1}{2}\Lambda_1 & 0 \\ \Gamma_i & \frac{\bar{u}_i^2}{2s^2} \end{bmatrix} \qquad i = 1, \ldots, n_u \\ & && 0 > \text{He} \begin{bmatrix} A\Lambda_1 & B\Lambda_2 & E & 0 \\ \Gamma + K\Lambda_1 & -\Lambda_2 + L\Lambda_2 & G & 0 \\ 0 & 0 & -\frac{1}{2}I & 0 \\ C\Lambda_1 & D\Lambda_2 & F & -\frac{\delta}{2}I \end{bmatrix} \end{aligned}$$

Feasibility for fixed $s > 0$ implies:

- The local $\mathcal{L}_2$-bound for $\gamma = \sqrt{\delta}$
- local asymptotic stablility for all $x \in \mathbb{R}^n$ contained in the sublevel set $\{x \in \mathbb{R}^n : x^T P x \leq s^2\}$.

$\mathcal{L}_2$-Stability and $\mathcal{L}_2$-Gain Optimization (Example)

We continue with the example:

$$\begin{aligned} \dot{x} &= Ax + B\,\mathrm{sat}(u) + Ew \\ z &= Cx + D\,\mathrm{sat}(u) + Fw \\ u &= Kx. \end{aligned}$$

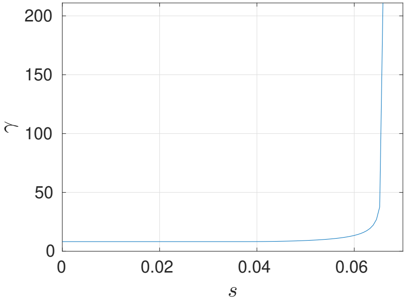
Using the deadzone operator:

$$\begin{aligned} \dot{x} &= Ax + BKx - Bq + Ew = (A + BK)x - Bq + Ew \\ z &= Cx + DBx - Dq + Fw = (C + DB)x - Dq + Fw \\ u &= Kx \\ q &= u - \mathrm{sat}(u) \end{aligned}$$

We continue with an earlier example (which we have stabilized in the unconstrained case):

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -20.93 & 21.92 & 11.83 \\ -15.62 & 15.28 & 8.22 \\ 4.31 & -4.64 & -2.61 \end{bmatrix} x + \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} q + \begin{bmatrix} 2 & -2 \\ 1 & 3 \\ 3 & -2 \end{bmatrix} w \\ z &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} -3 & 2 \end{bmatrix} w \\ u &= \begin{bmatrix} -7.31 & 6.64 & 4.61 \end{bmatrix} x + \begin{bmatrix} 2 & -3 \end{bmatrix} w \\ q &= u - \mathrm{sat}(u) \end{aligned}$$

Optimal $\mathcal{L}_2$-gain γ (with respect to s):



Section 4

Antiwindup Synthesis

Antiwindup Synthesis

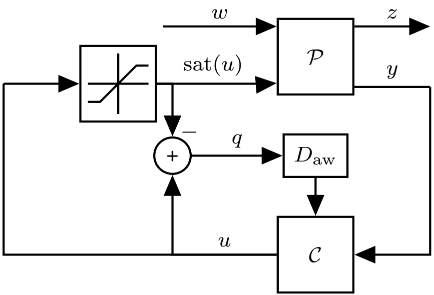
Plant & Controller:

$$\mathcal{P}:\begin{cases}\dot{x}_p &= A_p x_p + B_p\,\mathrm{sat}(u) + B_w w\\ y &= C_{p,y}x_p + D_{p,y}w\\ z &= C_{p,z}x_p + D_{p,z}w\end{cases}$$

$$\mathcal{C}:\begin{cases}\dot{x}_c &= A_c x_c + B_c y + D_{aw,1}q\\ u &= C_c x_c + D_{c,y}y + D_{aw,2}q\end{cases}$$

Anti-windup injection terms

- $D_{\mathrm{aw},1}$ and $D_{\mathrm{aw},2}$ are to be designed to improve the closed-loop performance.



Updated system dynamics:

$$\begin{aligned}\dot{x} &= Ax + (B + B_{\mathrm{aw}}D_{\mathrm{aw}})q + Ew\\ z &= Cx + Dq + Fw\\ u &= Kx + (L + L_{\mathrm{aw}}D_{\mathrm{aw}})q + Gw\\ q &= u - \mathrm{sat}(u)\end{aligned}$$

Design parameter:

$$D_{\mathrm{aw}} = \begin{bmatrix} D_{\mathrm{aw},1}\\ D_{\mathrm{aw},2}\end{bmatrix}$$

System/Controller parameter:

$$B_{\mathrm{aw}} = \begin{bmatrix} 0 & B_p\\ I_{n_c} & 0\end{bmatrix}, \qquad L_{\mathrm{aw}} = [0\ I_{n_u}].$$

Global Antiwindup Synthesis

Optimization problem:

$$
\begin{array}{lll}
\min\limits_{\Lambda_1,\Lambda_2,X,\gamma} & \gamma & \\
\text{subject to} & 0 < \Lambda_1 & \text{symmetric} \\
& 0 < \Lambda_2 & \text{diagonal} \\
& 0 < \gamma & \\
& 0 > \mathrm{He}\left[\begin{array}{ccc|c} A\Lambda_1 & B\Lambda_2 + B_{\mathrm{aw}}X & E & 0 \\ K\Lambda_1 & -\nu\Lambda_2 + L\Lambda_2 + L_{\mathrm{aw}}X & G & 0 \\ 0 & 0 & -\frac{\gamma}{2}I & 0 \\ \hline C\Lambda_1 & D & F & -\frac{\gamma}{2}I \end{array}\right] &
\end{array}
$$

$\rightsquigarrow$ If the optimization problem is feasible, the antiwindup injection term $D_{\mathrm{aw}} = X\Lambda_2^{-1}$

$\rightsquigarrow$ $\nu \in (0,1]$ can be used/decreased to obtain an implementable $D_{\mathrm{aw},2}$ (well-posedness of algebraic loop)

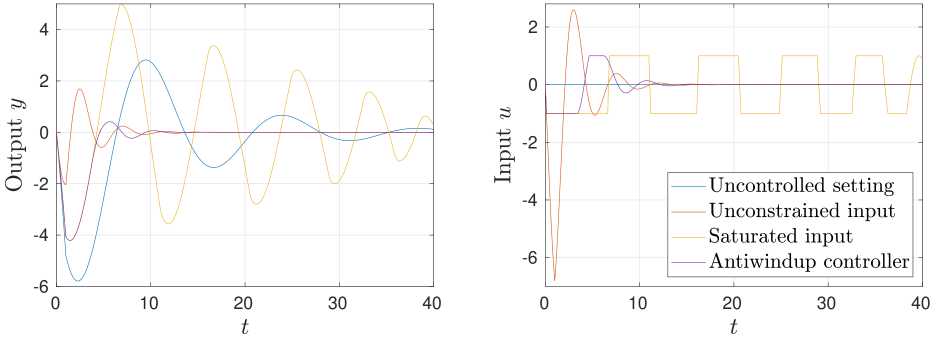
$\rightsquigarrow$ Local analysis can be performed using the same tricks discussed before

Global Antiwindup Synthesis (Example)

Consider the plant/controller defined through the dynamics (subject to the disturbances):

$$\left[\begin{array}{c|c|c} A_p & B_p & B_w \\ \hline C_{p,y} & & D_{p,y} \\ \hline C_{p,z} & & D_{p,z} \end{array}\right] = \left[\begin{array}{cc|c|c} -0.2 & -0.2 & 0.6 & 3 \\ 1 & 0 & 0.4 & 3 \\ \hline -0.4 & -0.9 & & 0 \\ \hline -0.4 & -0.9 & & 0 \end{array}\right], \qquad \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_{c,y} \end{array}\right] = \left[\begin{array}{c|c} 0 & 1 \\ \hline 2 & 2 \end{array}\right], \qquad w(t) = \begin{cases} 1, & \text{if } t \leq 1 \\ 0, & \text{if } t > 1 \end{cases}$$

$\rightsquigarrow D_{\text{aw},1} = -127.30$, $D_{\text{aw},2} = 0.45$.



Introduction to Nonlinear Control

Stability, control design, and estimation

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Part II:

Chapter 8: LMI Based Controller and Antiwindup Designs

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