Introduction to Nonlinear Control

Stability, control design, and estimation

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Part II:

Chapter 9: Control Lyapunov Functions

9.1 Control Affine Systems
9.2 ISS Redesign via L_gV Damping
9.3 Sontag's Universal Formula
9.4 Backstepping
9.5 Forwarding

Australian National University

Control Lyapunov Functions

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1. Control Affine Systems
2. ISS Redesign via L_gV Damping
3. Sontag's Universal Formula
4. Backstepping
   - Avoiding Cancellations
   - Convergence Structure
5. Forwarding
   - Convergence Structure

Control Lyapunov Functions

Consider the nonlinear system

$$\dot{x} = f(x, u)$$

- $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$
- state x and control input u
- Goal: Define a feedback control law $u = k(x)$ which asymptotically stabilizes the origin.

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Control Lyapunov function: $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$

- In terms of a feedback law $u = k(x)$,

  $$\tfrac{d}{dt} V(x(t)) = \langle \nabla V(x), f(x, k(x)) \rangle < 0, \qquad \forall\, x \neq 0$$

  $\rightsquigarrow V$ is a Lyapunov function for $\dot{x} = f(x, k(x)) = \tilde{f}(x)$
- For each $x \neq 0$ we can find u such that

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Definition (Control Lyapunov function (CLF))

Consider the nonlinear system and $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$. A continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is called control Lyapunov function for the system if

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|), \qquad \forall\, x \in \mathbb{R}^n,$$

and for all $x \in \mathbb{R}^n \backslash \{0\}$ there exists $u \in \mathbb{R}^m$ such that

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Section 1

Control Affine Systems

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$$\dot{x} = f(x) + g(x)u$$

Assumptions:

- for simplicity we focus on $u \in \mathbb{R}$
- $f, g : \mathbb{R}^n \to \mathbb{R}^n$ (locally Lipschitz)
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$$\langle v, a_1 v_1 + a_2 v_2 \rangle = a_1 \langle v, v_1 \rangle + a_2 \langle v, v_2 \rangle.$$

The decrease condition:

$$\begin{aligned}\dot{V}(x) &= \langle \nabla V(x), f(x) + g(x)u \rangle \\ &= L_f V(x) + L_g V(x) u < 0, \quad \forall\, x \neq 0.\end{aligned}$$

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The decrease condition for control affine systems:

$$L_f V(x) < 0 \quad \forall\ x \in \mathbb{R}^n \backslash \{0\} \ \text{ such that } \ L_g V(x) = 0$$

In other words

- If $L_g V(x) = 0$ (i.e., we have no control authority)
- then $L_f V(x) < 0$ needs to be satisfied

Section 2

ISS Redesign via L_gV Damping

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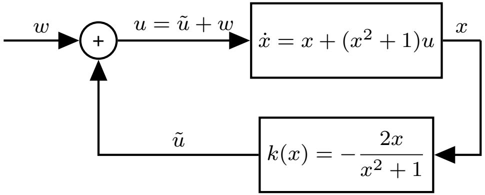
We begin with an example:

$$\dot{x} = x + (x^2 + 1)u.$$

One possible feedback stabilizer is given by

$$u = k(x) = -\frac{2x}{x^2 + 1}$$

which results in the closed-loop system $\dot{x} = -x$.



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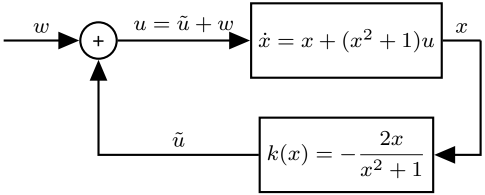

- Feedback law subject to disturbances

$$u = k(x) + w$$

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$$\dot{x} = -x + (x^2+1)w.$$

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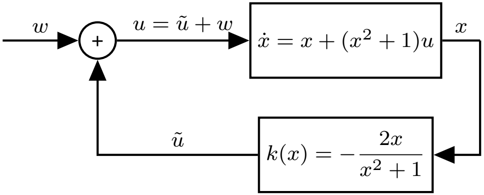

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- Suppose $V(x)$ is a Lyapunov function for $w = 0$ so that V satisfies

$$\dot{V}(x) = L_f V(x) + L_g V(x) k(x).$$

- Candidate ISS-Lyapunov function V would satisfy

$$\begin{aligned} \dot{V}(x) &= L_f V(x) + L_g V(x) k(x) + L_g V(x) w \\ &\leq L_f V(x) + L_g V(x) k(x) + \tfrac{1}{2} (L_g V(x))^2 + \tfrac{1}{2} w^2 \end{aligned}$$

- If the controller has an additional term of the form $-L_g V(x)$, this dominates $\frac{1}{2} (L_g V(x))^2$

⇝ $V(x)$ is an ISS-Lyapunov function.

ISS Redesign via $L_g V$ Damping (2)

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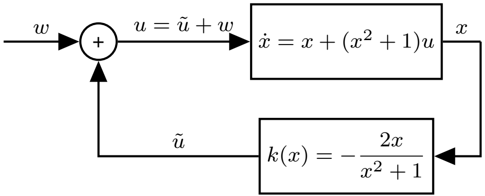

Back to the example:

- Take $V_1(x) = \frac{1}{2}x^2$ and augment controller:

$$u = k_1(x) = k(x) - L_g V_1(x) = -\frac{2x}{x^2 + 1} - x(x^2 + 1)$$

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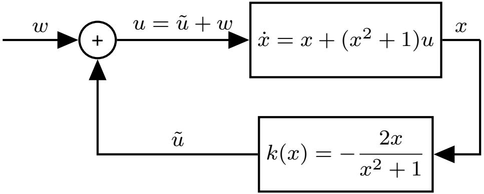

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- The closed-loop system with disturbance:

$$\dot{x} = -x - x(x^2 + 1)^2 + (x^2 + 1)w \doteq f_1(x, w).$$

- The new closed-loop system is ISS:

$$\begin{aligned}\langle \nabla V_1(x), f_1(x, w)\rangle &= -x^2 - x^2(x^2 + 1)^2 + x(x^2 + 1)w \\ &\leq -x^2 - \tfrac{1}{2}x^2(x^2 + 1)^2 + \tfrac{1}{2}w^2\end{aligned}$$

ISS Redesign via $L_g V$ Damping (General Procedure)

- Consider

$$\dot{x} = f(x) + g(x)k(x), \qquad u = k(x)$$

  (with asymptotically stable origin, i.e., stabilizing feedback $u = k(x)$)

- Corresponding Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ so that

$$L_f V(x) + L_g V(x)k(x) \leq -\alpha(|x|)$$

  where $\alpha \in \mathcal{K}_\infty$.

- Define the augmented feedback

$$u = k(x) - L_g V(x)$$

- Then, the system

$$\dot{x} = f(x) + g(x)\left(k(x) - L_g V(x)\right) + g(x)w$$

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- Indeed, V is an ISS-Lyapunov function:

$$\begin{aligned}
&\langle \nabla V(x), f(x) + g(x)\left(k(x) - L_gV(x)\right) + g(x)w\rangle \\
&\quad = L_fV(x) + L_gV(x)k(x) - \left(L_gV(x)\right)^2 + L_gV(x)w \\
&\quad \leq L_fV(x) + L_gV(x)k(x) - \tfrac{1}{2}\left(L_gV(x)\right)^2 + \tfrac{1}{2}w^2 \\
&\quad \leq -\alpha(|x|) + \tfrac{1}{2}w^2.
\end{aligned}$$

Note that: The method is also known as

- nonlinear damping
- Jurdjevic-Quinn controller.

Section 3

Sontag's Universal Formula

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Consider a control affine system ($u \in \mathbb{R}$)

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with corresponding CLF V, i.e.,

$$L_f V(x) < 0 \quad \forall \; x \in \mathbb{R}^n \backslash \{0\} \; \text{ such that } \; L_g V(x) = 0$$

Then, for $\kappa > 0$ define the feedback law

$$k(x) = \begin{cases} -\left(\kappa + \frac{L_f V(x) + \sqrt{L_f V(x)^2 + L_g V(x)^4}}{L_g V(x)^2}\right) L_g V(x), & L_g V(x) \neq 0 \\ 0, & L_g V(x) = 0 \end{cases}$$

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Sketch of the proof: For $\kappa = 0$ it holds that

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Note that: Formula known as

- Universal formula
- Sontag's formula

  (Derived by Eduardo Sontag)

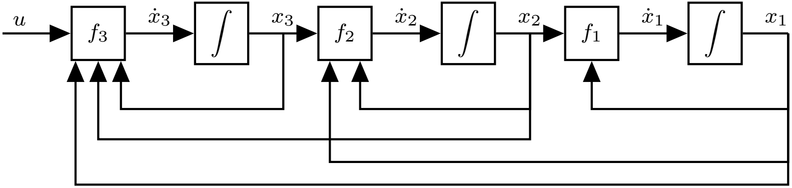
Section 4

Backstepping

Backstepping (How to find CLFs?)

Systems in *strict feedback form*:

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2)\\ \dot{x}_2 &= f_2(x_1, x_2, x_3)\\ &\vdots\\ \dot{x}_{n-1} &= f_{n-1}(x_1, x_2, \dots, x_{n-1}, x_n)\\ \dot{x}_n &= f_n(x_1, x_2, \dots, x_n, u).\end{aligned}$$



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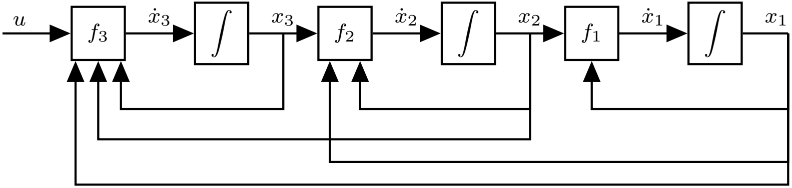

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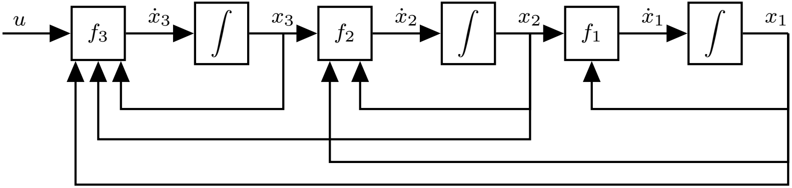

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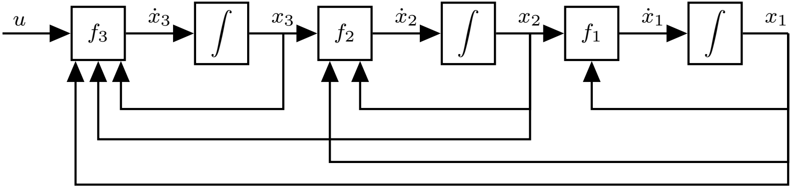

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- Actually the x-dynamics satisfy:

$$\dot{x} = x^3 + xk(x) - xk(x) + x\xi = -x^3 + x(\xi + 2x^2).$$

Backstepping (How to find CLFs?)

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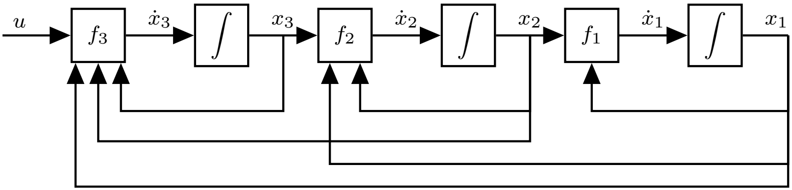
$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2)\\ \dot{x}_2 &= f_2(x_1, x_2, x_3)\\ &\vdots\\ \dot{x}_{n-1} &= f_{n-1}(x_1, x_2, \dots, x_{n-1}, x_n)\\ \dot{x}_n &= f_n(x_1, x_2, \dots, x_n, u).\end{aligned}$$



Backstepping idea (based on an example):

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Step 1: Define Virtual Control. Suppose that ξ is a control input for the x-subsystem (i.e., ξ as a virtual control for x)

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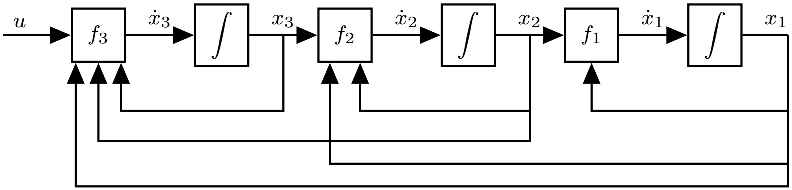

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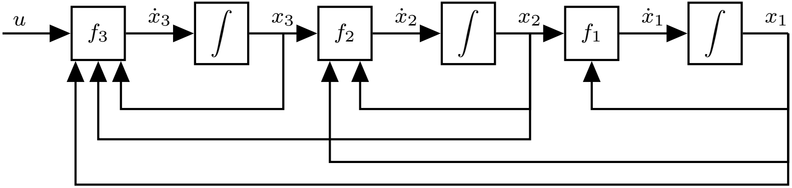

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- System in (x, z) coordinates:

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Step 3: Construct a Control Lyapunov Function.

$$V_a(x, z) = V(x) + \tfrac{1}{2}z^2 = \tfrac{1}{2}x^2 + \tfrac{1}{2}z^2.$$

It holds that

$$\dot{V}_a(x, z) = -x^4 + x^2 z + z(u - 4x^4 + 4x^2 z)$$
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$\rightsquigarrow$ The derivative is negative for u appropriate

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Step 4: Construct a feedback stabilizer. Define (for example)

$$u = k_1(x, z) = -x^2 + 4x^4 - 4x^2 z - z$$

then

$$\dot{V}_a(x, z) = -x^4 - z^2$$

In the original variables:

$$u = -x^2 + 4x^4 - 4x^2(\xi + 2x^2) - (\xi + 2x^2)$$

Backstepping (How to find CLFs?) (3)

Backstepping idea (based on an example):

$$\dot{x} = x^3 + x\xi = -x^3 + x(\xi + 2x^2)$$
$$\dot{\xi} = u.$$

Introduce error dynamics

$$z = \xi - k(x) = \xi + 2x^2$$

System in (x, z) coordinates:

$$\dot{x} = -x^3 + xz$$
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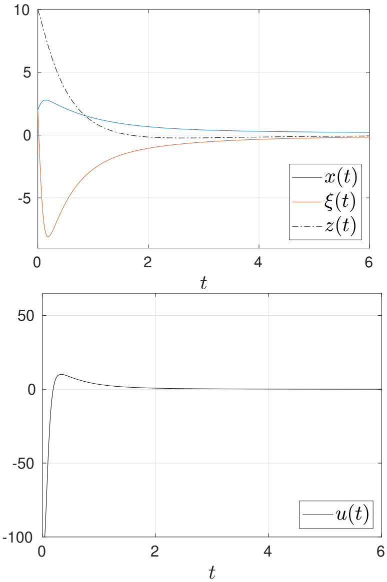

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Error dynamics

$$\begin{bmatrix}\dot{z}_0\\ \dot{z}_1\\ \vdots\\ \dot{z}_i\end{bmatrix} = \begin{bmatrix}\tilde{f}_1(z_0, k_1(z_0))\\ \tilde{f}_2(z_0, z_1, k_2(z_0, z_1))\\ \vdots\\ \tilde{f}_{i+1}(z_0, z_1, \ldots, z_{i-1}, x_{i+1})\end{bmatrix}$$

for $i = 1, \ldots, n$, is used.

Input: Define $z_0 = x_1$, $x_{n+1} = u$, $\tilde{f}_1 = f_1$ and, $V_0 = 0$.
Output: Stabilizing feedback law u.
For $i = 1, 2, \ldots, n$

1. Consider error dynamics & virtual control $x_{i+1} = k_i(z_0, \ldots, z_{i-1})$
2. Define k_i in such a way that the origin of the error dynamics is asymptotically stable and define $\tilde{V}_i(z_0, \ldots, z_{i-1})$ so that

$$V_i(z_0, \ldots, z_{i-1}) \doteq V_{i-1}(z_0, \ldots, z_{i-2}) + \tilde{V}_i(z_0, \ldots, z_{i-1})$$

is a Lyapunov function.

3. If $i \neq n$, define the error dynamics $z_i = x_{i+1} - k_i(z_0, \ldots, z_{i-1})$ with

$$\begin{aligned}\dot{z}_i &= \dot{x}_{i+1} - \tfrac{d}{dt}k_i(z_0, \ldots, z_{i-1})\\ &= f_{i+1}(x_1, \ldots, x_{i+1}) - \tfrac{d}{dt}k_i(z_0, \ldots, z_{i-1})\\ &= \tilde{f}_{i+1}(z_0, \ldots, z_{i-1}, x_{i+1}).\end{aligned}$$

4. If $i = n$ return the input

$$u(x_1, \ldots, x_n) \doteq k_n(z_0, \ldots, z_{n-1})$$

and the CLF $V(x_1, \ldots, x_n) \doteq V_n(z_0, \ldots, z_{n-1})$.

Backstepping: Avoiding Cancellations

Consider:

$$\dot{x} = f(x) + g(x)\xi$$
$$\dot{\xi} = u.$$

Virtual stabilizing feedback $\xi = k(x)$ & error variable $z = \xi - k(x)$:

$$\dot{x} = f(x) + g(x)k(x) + g(x)z$$
$$\dot{z} = u - \tfrac{\partial}{\partial x}k(x)\dot{x}.$$

Feedback derived on previous slides :

$$u(x,z) = -L_g V(x) + \tfrac{\partial}{\partial x}k(x)\left(f(x) + g(x)(k(x)+z)\right) - z$$

(Based on $V_a(x,z) = V(x) + \frac{1}{2}z^2$)

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Time derivative:

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Cancelling

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Feedback stabilizer

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Cancelling

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Note that

- Simpler feedback
- More complicated CLF

Backstepping: Avoiding Cancellations (2)

Recall the example:

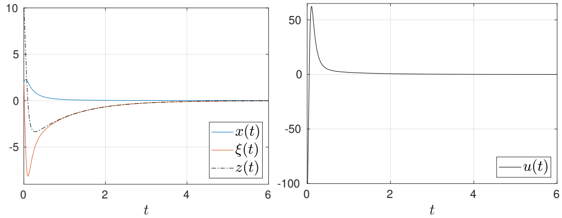
$$\dot{x} = x^3 + x\xi = -x^3 + x(\xi + 2x^2)$$
$$\dot{\xi} = u.$$

Error dynamics

$$z = \xi - k(x) = \xi + 2x^2$$

CLF and feedback law: (avoiding cancellation)

$$V_a(x,z) = \tfrac{1}{2}x^2 + x^4 + \tfrac{1}{2}z^2$$
$$u(x,z) = -x^2 - 4x^2 z - z.$$



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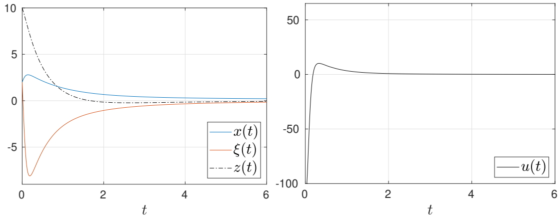

Exact Backstepping and a High-Gain Alternative

Consider the example with an additional integrator

$$\dot{x} = x^3 + x\xi_1, \qquad \dot{\xi}_1 = \xi_2, \qquad \dot{\xi}_2 = u$$

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So far, we have defined: $\xi_1 = k_1(x) = -2x^2$

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Error dynamics and CLF:

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$$V(x, z_1) = \tfrac{1}{2}x^2 + x^4 + \tfrac{1}{2}z_1^2$$

We continue with

$$\xi_2 = k_2(x, z_1) = -x^2 - 4x^2 z_1 - z_1.$$

Define the error variable $z_2 = \xi_2 - k_2(x, z_1)$ so that

$$\dot{x} = -x^3 + xz_1$$

$$\dot{z}_1 = z_2 + k_2(x, z_1) - 4x^4 + 4x^2 z_1 = -z_1 + z_2 - x^2 - 4x^4$$

$$\dot{z}_2 = u - \overbrace{k_2(x, z_1)}^{\cdot}.$$

We continue

$$\begin{aligned}\overbrace{k_2(x, z_1)}^{\cdot} &= (-8xz_1 - 2x)\dot{x} + (-4x^2 - 1)\dot{z}_1 \\ &= (-8xz_1 - 2x)(-x^3 + xz_1) \\ &\quad + (-4x^2 - 1)(z_2 - x^2 - z_1 - 4x^4).\end{aligned}$$

The CLF (extending the previous one)

$$V(x, z_1, z_2) = \tfrac{1}{2}x^2 + x^4 + \tfrac{1}{2}z_1^2 + \tfrac{1}{2}z_2^2$$

naturally leads to ($\rightsquigarrow$ exact backstepping)

$$u = -z_1 - z_2 + \overbrace{k_2(x, z_1)}^{\cdot}$$

Exact Backstepping and a High-Gain Alternative

Consider the example with an additional integrator

$$\dot{x} = x^3 + x\xi_1, \qquad \dot{\xi}_1 = \xi_2, \qquad \dot{\xi}_2 = u$$

So far, we have defined: $\xi_1 = k_1(x) = -2x^2$

$$z_1 = \xi_1 - k_1(x) = \xi_1 + 2x^2$$

Error dynamics and CLF:

$$\dot{x} = -x^3 + xz_1$$

$$\dot{z}_1 = \xi_2 - \frac{\partial k_1}{\partial x}(x)\left(-x^3 + xz_1\right) = \xi_2 - 4x^4 + 4x^2 z_1$$

$$V(x, z_1) = \tfrac{1}{2}x^2 + x^4 + \tfrac{1}{2}z_1^2$$

We continue with

$$\xi_2 = k_2(x, z_1) = -x^2 - 4x^2 z_1 - z_1.$$

Define the error variable $z_2 = \xi_2 - k_2(x, z_1)$ so that

$$\dot{x} = -x^3 + xz_1$$

$$\dot{z}_1 = z_2 + k_2(x, z_1) - 4x^4 + 4x^2 z_1 = -z_1 + z_2 - x^2 - 4x^4$$

$$\dot{z}_2 = u - \overset{\cdot}{\overbrace{k_2(x, z_1)}}.$$

We continue

$$\overset{\cdot}{\overbrace{k_2(x, z_1)}} = (-8xz_1 - 2x)\dot{x} + (-4x^2 - 1)\dot{z}_1$$
$$= (-8xz_1 - 2x)(-x^3 + xz_1)$$
$$+ (-4x^2 - 1)(z_2 - x^2 - z_1 - 4x^4).$$

The CLF (extending the previous one)

$$V(x, z_1, z_2) = \tfrac{1}{2}x^2 + x^4 + \tfrac{1}{2}z_1^2 + \tfrac{1}{2}z_2^2$$

naturally leads to ($\rightsquigarrow$ exact backstepping)

$$u = -z_1 - z_2 + \overset{\cdot}{\overbrace{k_2(x, z_1)}}$$

As an alternative:

- Instead of cancelling $\dot{k}_1$ dominate it with a linear term
- In other words, consider the virtual control

$$\xi_2 = -\kappa z_1, \qquad \kappa > 0$$

Exact Backstepping and a High-Gain Alternative (2)

Consider the example with an additional integrator

$$\dot{x} = x^3 + x\xi_1, \qquad \dot{\xi}_1 = \xi_2, \qquad \dot{\xi}_2 = u$$

So far, we have defined: $\xi_1 = k_1(x) = -2x^2$

$$z_1 = \xi_1 - k_1(x) = \xi_1 + 2x^2$$

Error dynamics and CLF:

$$\dot{x} = -x^3 + xz_1$$

$$\dot{z}_1 = \xi_2 - \frac{\partial k_1}{\partial x}(x)\left(-x^3 + xz_1\right) = \xi_2 - 4x^4 + 4x^2 z_1$$

$$V(x, z_1) = \frac{1}{2}x^2 + x^4 + \frac{1}{2}z_1^2$$

Exact Backstepping and a High-Gain Alternative (2)

Consider the example with an additional integrator

$$\dot{x} = x^3 + x\xi_1, \qquad \dot{\xi}_1 = \xi_2, \qquad \dot{\xi}_2 = u$$

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$$\dot{z}_1 = \xi_2 - \frac{\partial k_1}{\partial x}(x)\left(-x^3 + xz_1\right) = \xi_2 - 4x^4 + 4x^2 z_1$$

$$V(x, z_1) = \tfrac{1}{2}x^2 + x^4 + \tfrac{1}{2}z_1^2$$

Consider virtual control

$$\xi_2 = -\kappa z_1, \qquad \kappa > 0$$

We have

$$\dot{x} = -x^3 + xz_1$$

$$\dot{z}_1 = -\kappa z_1 - 4x^4 + 4x^2 z_1.$$

CLF

$$V(x, z_1, z_2) = \tfrac{1}{2}x^2 + x^4 + \tfrac{1}{2}z_1^2 + \tfrac{1}{2}z_2^2$$

Then

$$\begin{aligned}
&\dot{V}(x, z_1) \\
&= -x^4 + x^2 z_1 - 4x^6 + 4x^4 z_1 - \kappa z_1^2 - 4x^4 z_1 + 4x^2 z_1^2 \\
&\leq -x^4 - 4x^6 + \tfrac{1}{2}x^4 + \tfrac{1}{2}z_1^2 - \kappa z_1^2 + 4x^2 z_1^2 \\
&= -\tfrac{1}{2}x^4 - 4x^6 - z_1^2\left(\kappa - \tfrac{1}{2} - 4x^2\right)
\end{aligned}$$

Exact Backstepping and a High-Gain Alternative (2)

Consider the example with an additional integrator

$$\dot{x} = x^3 + x\xi_1, \qquad \dot{\xi}_1 = \xi_2, \qquad \dot{\xi}_2 = u$$

So far, we have defined: $\xi_1 = k_1(x) = -2x^2$

$$z_1 = \xi_1 - k_1(x) = \xi_1 + 2x^2$$

Error dynamics and CLF:

$$\dot{x} = -x^3 + xz_1$$
$$\dot{z}_1 = \xi_2 - \frac{\partial k_1}{\partial x}(x)\left(-x^3 + xz_1\right) = \xi_2 - 4x^4 + 4x^2 z_1$$
$$V(x, z_1) = \tfrac{1}{2}x^2 + x^4 + \tfrac{1}{2}z_1^2$$

Consider virtual control

$$\xi_2 = -\kappa z_1, \qquad \kappa > 0$$

We have

$$\dot{x} = -x^3 + xz_1$$
$$\dot{z}_1 = -\kappa z_1 - 4x^4 + 4x^2 z_1.$$

CLF

$$V(x, z_1, z_2) = \tfrac{1}{2}x^2 + x^4 + \tfrac{1}{2}z_1^2 + \tfrac{1}{2}z_2^2$$

Then

$$\begin{aligned}
&\dot{V}(x, z_1) \\
&= -x^4 + x^2 z_1 - 4x^6 + 4x^4 z_1 - \kappa z_1^2 - 4x^4 z_1 + 4x^2 z_1^2 \\
&\leq -x^4 - 4x^6 + \tfrac{1}{2}x^4 + \tfrac{1}{2}z_1^2 - \kappa z_1^2 + 4x^2 z_1^2 \\
&= -\tfrac{1}{2}x^4 - 4x^6 - z_1^2\left(\kappa - \tfrac{1}{2} - 4x^2\right)
\end{aligned}$$

Therefore, if

$$\kappa > \tfrac{1}{2} + 4x^2 \quad \text{or equivalently} \quad x^2 < \frac{\kappa - \frac{1}{2}}{4}$$

then the origin is locally asymptotically stable
$\rightsquigarrow$ Increasing κ, increases the region of attraction

Exact Backstepping and a High-Gain Alternative (2)

Consider the example with an additional integrator

$$\dot{x} = x^3 + x\xi_1, \qquad \dot{\xi}_1 = \xi_2, \qquad \dot{\xi}_2 = u$$

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$$V(x, z_1) = \tfrac{1}{2}x^2 + x^4 + \tfrac{1}{2}z_1^2$$

Consider virtual control

$$\xi_2 = -\kappa z_1, \qquad \kappa > 0$$

We have

$$\dot{x} = -x^3 + xz_1$$

$$\dot{z}_1 = -\kappa z_1 - 4x^4 + 4x^2 z_1.$$

CLF

$$V(x, z_1, z_2) = \tfrac{1}{2}x^2 + x^4 + \tfrac{1}{2}z_1^2 + \tfrac{1}{2}z_2^2$$

Then

$$\begin{aligned}
&\dot{V}(x, z_1) \\
&= -x^4 + x^2 z_1 - 4x^6 + 4x^4 z_1 - \kappa z_1^2 - 4x^4 z_1 + 4x^2 z_1^2 \\
&\leq -x^4 - 4x^6 + \tfrac{1}{2}x^4 + \tfrac{1}{2}z_1^2 - \kappa z_1^2 + 4x^2 z_1^2 \\
&= -\tfrac{1}{2}x^4 - 4x^6 - z_1^2\left(\kappa - \tfrac{1}{2} - 4x^2\right)
\end{aligned}$$

Therefore, if

$$\kappa > \tfrac{1}{2} + 4x^2 \quad \text{or equivalently} \quad x^2 < \frac{\kappa - \frac{1}{2}}{4}$$

then the origin is locally asymptotically stable
$\rightsquigarrow$ Increasing κ, increases the region of attraction

Subsequent step: Let $z_2 = \xi_2 + \kappa z_1$. Then

$$\dot{x} = -x^3 + xz_1$$

$$\dot{z}_1 = -\kappa z_1 - 4x^4 - 4x^2 z_1 + z_2$$

$$\dot{z}_2 = u + \kappa\left(-\kappa z_1 - 4x^4 - 4x^2 z_1 + z_2\right).$$

We again use a dominating linear term $u = -\kappa z_2$ which leads to

$$u = -\kappa\left(\xi_2 + \kappa(\xi_1 + 2x^2)\right)$$

Exact Backstepping and a High-Gain Alternative (3)

Theorem (High-gain backstepping)

Consider the system

$$\begin{aligned} \dot{x} &= f(x) + g(x)\xi_1 \\ \dot{\xi}_1 &= \xi_2 \\ &\vdots \\ \dot{\xi}_n &= u \end{aligned}$$

in strict feedback form, let $\kappa \in \mathbb{R}_{>0}$ *be a design parameter and assume there exists a feedback stabilizer* $\xi_1 = k(x)$ *and an associated control Lyapunov function* $V(x)$. *Let*

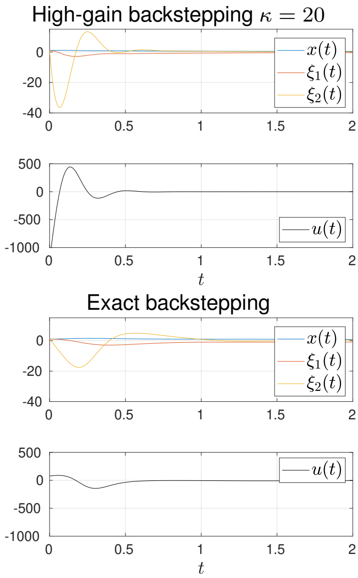
$$p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

be an arbitrary Hurwitz polynomial. Then the feedback

$$u = -\kappa\left(a_{n-1}\xi_n + \kappa\left(a_{n-2}\xi_{n-1} + \kappa\left(\cdots + \kappa(a_1\xi_2 + \kappa a_0(\xi_1 - k(x)))\cdots\right)\right)\right)$$

achieves semiglobal stabilization of $[x^T, \xi^T]^T = 0$.

(Semiglobal refers to the fact that we have a design parameter, κ, which can be tuned to make the region of attraction for the origin as large as we wish.)



Backstepping: Convergence Structure

Consider again the example:

$$\dot{x} = x^3 + x\xi$$
$$\dot{\xi} = u$$

with error dynamics

$$z = \xi - k(x) = \xi + 2x^2$$

Exact backstepping:

$$u(x,\xi) = -x^2 + 4x^4 - 4x^2(\xi + 2x^2) - (\xi + 2x^2)$$

High-gain backstepping: ($\kappa > 0$, $p(\lambda) = \lambda + 1$)

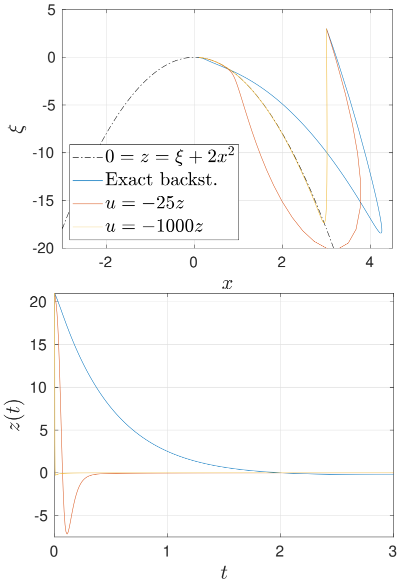
$$u(x,\xi) = -\kappa^2(\xi + 2x^2) = -\kappa^2 z$$

using high-gain backstepping.
The set where z vanishes:

$$\mathcal{Z} := \{[x,\xi]^T \in \mathbb{R}^2 : 0 = \xi - 2x^2\}$$

For large $\kappa > 0$ we observe two phases:

- convergence to $\mathcal{Z}$
- slide along $\mathcal{Z}$ to the origin



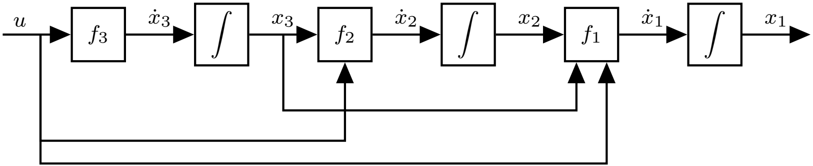
Section 5

Forwarding

Forwarding

Strict feedforward form:

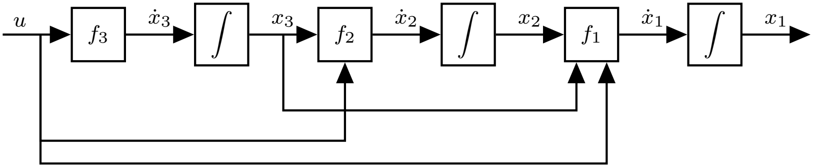
$$
\begin{aligned}
\dot{x}_1 &= f_1(x_2, x_3, \ldots, x_n, u) \\
\dot{x}_2 &= f_2(x_3, x_4, \ldots, x_n, u) \\
&\vdots \\
\dot{x}_{n-1} &= f_{n-1}(x_n, u) \\
\dot{x}_n &= f_n(u)
\end{aligned}
$$



Forwarding

Strict feedforward form:

$$\begin{aligned}\dot{x}_1 &= f_1(x_2, x_3, \ldots, x_n, u)\\ \dot{x}_2 &= f_2(x_3, x_4, \ldots, x_n, u)\\ &\vdots\\ \dot{x}_{n-1} &= f_{n-1}(x_n, u)\\ \dot{x}_n &= f_n(u)\end{aligned}$$



To introduce the idea consider:

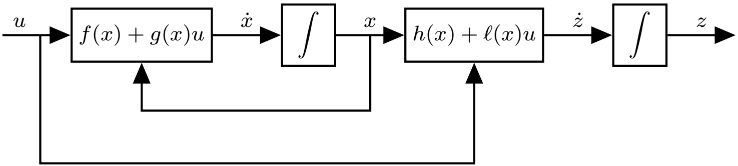
$$\begin{aligned}\dot{z} &= h(x) + \ell(x)u\\ \dot{x} &= f(x) + g(x)u\end{aligned}$$

(System in feedforward form)

Suppose that

- For $\dot{x} = f(x)$, 0 is asympt. stable
- V is a corresponding Lyap. fcn
- $\mathcal{M}(x)$ is a solution to the partial differential equation ($\mathcal{M}(0) = 0$)

$$L_f \mathcal{M}(x) = \langle \nabla \mathcal{M}(x), f(x) \rangle = h(x)$$



Forwarding (2)

To introduce the idea consider:

$$\dot{z} = h(x) + \ell(x)u$$
$$\dot{x} = f(x) + g(x)u$$

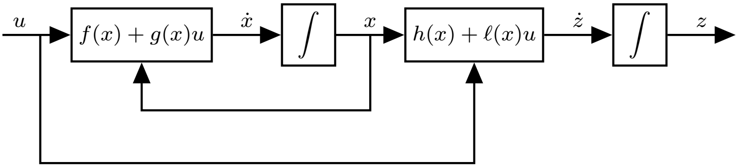
Suppose that

- For $\dot{x} = f(x)$, 0 is asympt. stable
- V is a corresponding Lyap. fcn
- $\mathcal{M}(x)$ is a solution to the partial differential equation ($\mathcal{M}(0) = 0$)

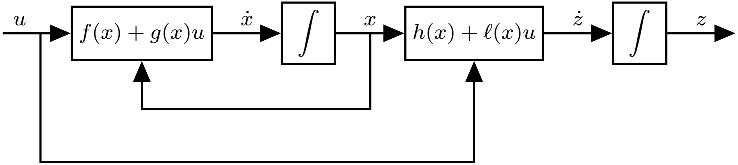
  $$L_f\mathcal{M}(x) = \langle \nabla\mathcal{M}(x), f(x)\rangle = h(x)$$

- If we are able to find a solution to the PDE with $\ell(0) - L_g\mathcal{M}(0) \neq 0$
- Then a CLF for the overall system is given by

  $$W(x,z) = V(x) + \tfrac{1}{2}\left(z - \mathcal{M}(x)\right)^2$$



Forwarding (2)



To introduce the idea consider:

$$\dot{z} = h(x) + \ell(x)u$$
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Suppose that

- For $\dot{x} = f(x)$, 0 is asympt. stable
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- If we are able to find a solution to the PDE with $\ell(0) - L_g\mathcal{M}(0) \neq 0$
- Then a CLF for the overall system is given by

  $$W(x,z) = V(x) + \tfrac{1}{2}\left(z - \mathcal{M}(x)\right)^2$$

Indeed, the time derivative of W yields:

$$\dot{W}(x,z) = L_fV(x) + L_gV(x)u + (z - \mathcal{M}(x))\left(\dot{z} - L_f\mathcal{M}(x) - L_g\mathcal{M}(x)u\right)$$
$$= L_fV(x) + L_gV(x)u + (z - \mathcal{M}(x))\left(h(x) + \ell(x)u - L_f\mathcal{M}(x) - L_g\mathcal{M}(x)u\right)$$
$$= L_fV(x) + L_gV(x)u + (z - \mathcal{M}(x))\left(\ell(x)u - L_g\mathcal{M}(x)u\right)$$
$$= L_fV(x) + u\left[L_gV(x) + (z - \mathcal{M}(x))\left(\ell(x) - L_g\mathcal{M}(x)\right)\right]$$

Forwarding (2)

To introduce the idea consider:

$$\dot{z} = h(x) + \ell(x)u$$
$$\dot{x} = f(x) + g(x)u$$

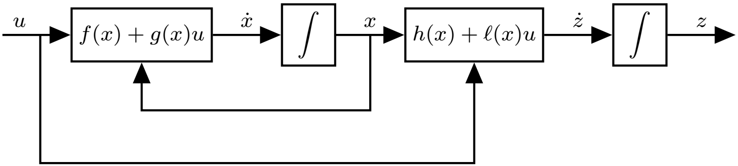
Suppose that

- For $\dot{x} = f(x)$, 0 is asympt. stable
- V is a corresponding Lyap. fcn
- $\mathcal{M}(x)$ is a solution to the partial differential equation ($\mathcal{M}(0) = 0$)

  $$L_f\mathcal{M}(x) = \langle \nabla\mathcal{M}(x), f(x)\rangle = h(x)$$

- If we are able to find a solution to the PDE with $\ell(0) - L_g\mathcal{M}(0) \neq 0$
- Then a CLF for the overall system is given by

  $$W(x, z) = V(x) + \tfrac{1}{2}\left(z - \mathcal{M}(x)\right)^2$$



Indeed, the time derivative of W yields:

$$\begin{aligned}\dot{W}(x,z) &= L_fV(x) + L_gV(x)u + (z - \mathcal{M}(x))\left(\dot{z} - L_f\mathcal{M}(x) - L_g\mathcal{M}(x)u\right)\\ &= L_fV(x) + L_gV(x)u + (z - \mathcal{M}(x))\left(h(x) + \ell(x)u - L_f\mathcal{M}(x) - L_g\mathcal{M}(x)u\right)\\ &= L_fV(x) + L_gV(x)u + (z - \mathcal{M}(x))\left(\ell(x)u - L_g\mathcal{M}(x)u\right)\\ &= L_fV(x) + u\left[L_gV(x) + (z - \mathcal{M}(x))\left(\ell(x) - L_g\mathcal{M}(x)\right)\right]\end{aligned}$$

Note that:

- The condition $\ell(0) - L_g\mathcal{M}(0) \neq 0$ is required to guarantee a decrease in z.

Possible feedback law: ($\kappa > 0$ design parameter)

$$u = -\kappa\left(L_gV(x) + (z - \mathcal{M}(x))\left(\ell(x) - L_g\mathcal{M}(x)\right)\right)$$

Forwarding (3)

Theorem

Consider the dynamical system and let $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a continuously differentiable positive definite Lyapunov function for $\dot{x} = f(x)$. Suppose there exists a solution $\mathcal{M} : \mathbb{R}^n \to \mathbb{R}^n$ to the PDE such that $\ell(0) - L_g\mathcal{M}(0) \neq 0$. Then W is a control Lyapunov function of the overall system and u is a globally asymptotically stabilizing feedback law.

System dynamics

$$\dot{z} = h(x) + \ell(x)u$$
$$\dot{x} = f(x) + g(x)u$$

Partial differential equation (PDE):

$$L_f\mathcal{M}(x) = \langle \nabla\mathcal{M}(x), f(x)\rangle = h(x), \quad \mathcal{M}(0) = 0$$

Control Lyapunov function:

$$W(x,z) = V(x) + \tfrac{1}{2}\left(z - \mathcal{M}(x)\right)^2$$

Feedback law: ($\kappa > 0$ design parameter)

$$u = -\kappa\left(L_gV(x) + (z - \mathcal{M}(x))\left(\ell(x) - L_g\mathcal{M}(x)\right)\right)$$

Forwarding (Example)

Consider

$$\dot z = x - x^2 u$$
$$\dot x = u.$$

Modify the input:

$$u = -x + v$$

then

$$\dot z = x - x^2(-x+v) = (x + x^3) + (-x^2)v = h(x) + \ell(x)v$$
$$\dot x = -x + v = f(x) + g(x)v.$$

Lyapunov function for $\dot x = -x$:

$$V(x) = \tfrac{1}{2}x^2$$

PDE: (unknown $\mathcal{M}(x)$ with $\mathcal{M}(0) = 0$, $\ell(0) - L_g\mathcal{M}(0) \neq 0$)

$$h(x) = \frac{\partial \mathcal{M}(x)}{\partial x} f(x), \quad \text{i.e.,} \quad x + x^3 = \frac{\partial \mathcal{M}(x)}{\partial x}(-x).$$

Thus

$$\mathcal{M}(x) = -\tfrac{1}{3}x^3 - x, \quad \text{with} \quad \ell(0) - L_g\mathcal{M}(0) = -1 \neq 0$$

Therefore, a control Lyapunov function is given by

$$W(x,z) = \tfrac{1}{2}x^2 + \tfrac{1}{2}\left(z + x + \tfrac{1}{3}x^3\right)^2.$$

Indeed,

$$\begin{aligned}
\dot W(x,z) &= -x^2 + xv + \left(z + x + \tfrac{1}{3}x^3\right)\left(\dot z + \dot x + x^2 \dot x\right) \\
&= -x^2 + xv + \left(z + x + \tfrac{1}{3}x^3\right) \\
&\qquad \cdot \left(x + x^3 - x^2 v - x + v - x^3 + x^2 v\right) \\
&= -x^2 + xv + \left(z + x + \tfrac{1}{3}x^3\right) v \\
&= -x^2 + \left(z + 2x + \tfrac{1}{3}x^3\right) v.
\end{aligned}$$

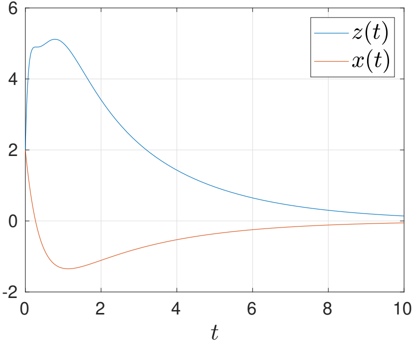
We choose the feedback stabilizer

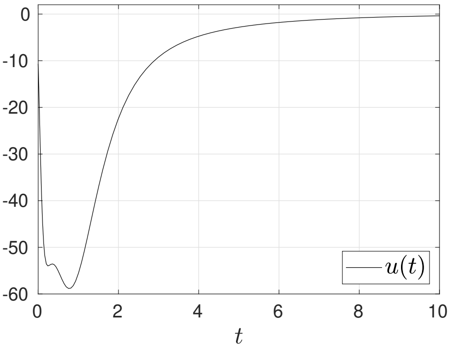
$$v = -\left(z + 2x + \tfrac{1}{3}x^3\right)$$

Hence, the control law in terms of u is given by:

$$u = -x + v = -z - 3x + \tfrac{1}{3}x^3.$$

Forwarding (Example, 2)





Forwarding: Recursive Application

Consider

$$\begin{aligned} \dot{z}_2 &= h_2(x, z_1) + \ell_2(x, z_1)u \\ \dot{z}_1 &= h_1(x) + \ell_1(x)u \\ \dot{x} &= f(x) + g(x)u \end{aligned}$$

Note that

- We have seen how to construct a CLF for (x, z_1)
- Once we have a CLF for the (x, z_1) dynamics we can relabel x as (x, z_1) and z as z_2 and apply the forwarding procedure again

Forwarding: Convergence Structure

Time derivative of the CLF W:

$$\dot{W}(x,z) = L_f V(x) + u\left(L_g V(x) + (z - \mathcal{M}(x))\left(\ell(x) - L_g\mathcal{M}(x)\right)\right)$$

Feedback law ($\kappa = 1$):

$$u = -\left(L_g V(x) + (z - \mathcal{M}(x))\left(\ell(x) - L_g\mathcal{M}(x)\right)\right)$$

- In addition, assume that $x = 0$ is asymptotically stable for $\dot{x} = f(x)$ i.e., $L_f V(x) < 0$ for all $x \neq 0$ and $u = 0$.
- $u(x,z) = 0$ is satisfied on the *forwarding manifold*

$$\left\{(x,z) \in \mathbb{R}^{n+m} : \ z = \mathcal{M}(x) + \tfrac{L_g V(x)}{\ell(x) - L_g\mathcal{M}(x)}\right\}.$$

⇝ u thus drives the system to the forwarding manifold

⇝ $L_f V(x) < 0$ for all $x \neq 0$ guarantees convergence to the origin (once (x,z) is close to the forwarding manifold)

Forwarding: Convergence Structure

Time derivative of the CLF W:

$$\dot{W}(x,z) = L_f V(x) + u\left(L_g V(x) + (z - \mathcal{M}(x))\left(\ell(x) - L_g \mathcal{M}(x)\right)\right)$$

Feedback law ($\kappa = 1$):

$$u = -\left(L_g V(x) + (z - \mathcal{M}(x))\left(\ell(x) - L_g \mathcal{M}(x)\right)\right)$$

- In addition, assume that $x = 0$ is asymptotically stable for $\dot{x} = f(x)$ i.e., $L_f V(x) < 0$ for all $x \neq 0$ and $u = 0$.
- $u(x,z) = 0$ is satisfied on the *forwarding manifold*

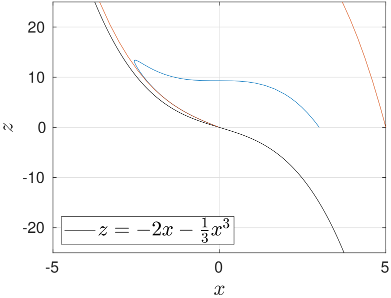
$$\left\{(x,z) \in \mathbb{R}^{n+m} : \; z = \mathcal{M}(x) + \frac{L_g V(x)}{\ell(x) - L_g \mathcal{M}(x)}\right\}.$$

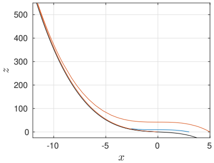
⇝ u thus drives the system to the forwarding manifold

⇝ $L_f V(x) < 0$ for all $x \neq 0$ guarantees convergence to the origin (once (x,z) is close to the forwarding manifold)

Recall

- $\mathcal{M}(x) = -\frac{1}{3}x^3 - x$
- $v(x,z) = (-z + 2x + \frac{1}{3}x^3)$
- $\dot{z} = x + x^3 - x^2 v, \qquad \dot{x} = -x + v$
- The forwarding manifold is defined through $z = -2x - \frac{1}{3}x^3$





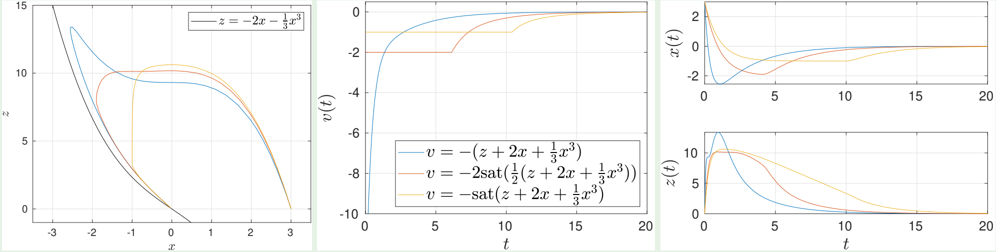
Forwarding: Saturated Control

- Maintain the assumption that the origin is asymptotically stable for $\dot{x} = f(x)$.
- Then, note that

$$u = -c \cdot \text{sat}\left(\frac{1}{c}\left(L_g V(x) + (z - \mathcal{M}(x))\left(\ell(x) - L_g \mathcal{M}(x)\right)\right)\right) \qquad \text{guarantees } \dot{W}(x,z) < 0,\ (x,z) \neq 0, \forall\, c > 0$$

  guarantees $\dot{W}(x,z) < 0$, $(x,z) \neq 0$, for all values of $c > 0$.
- Note that $u \in [-c, c]$ and still guarantees asymptotic stability of the origin (under the assumption on f)

Example (Back to the example ($c \in \{1,2\}$))



Note that: v is bounded! However, $u = -x + v$ is not bounded!

Introduction to Nonlinear Control

Stability, control design, and estimation

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Part II:

Chapter 9: Control Lyapunov Functions

9.1 Control Affine Systems
9.2 ISS Redesign via L_gV Damping
9.3 Sontag's Universal Formula
9.4 Backstepping
9.5 Forwarding

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