Introduction to Nonlinear Control

Stability, control design, and estimation

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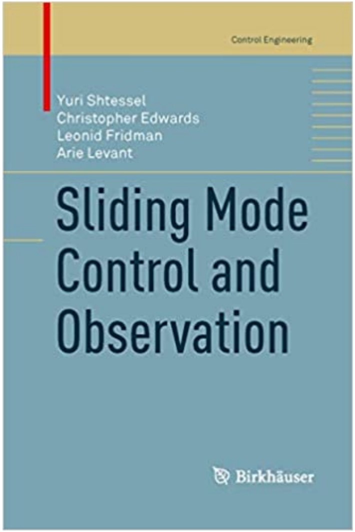
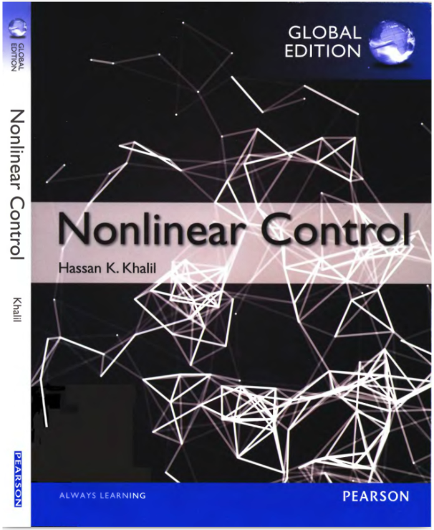
Part II:

Chapter 10: Sliding Mode Control

Australian National University

Sliding Mode Control





Nonlinear Systems - Fundamentals

Sliding Mode Control

We consider systems of the form

$$\dot{x} = f(x, u, \delta(t, x))$$
$$y = h(x)$$

with

- state $x \in \mathbb{R}^n$
- input $u \in \mathbb{R}^m$
- output $y \in \mathbb{R}$
- potentially time and state dependent unknown disturbance $\delta : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$

We will be interested in

- stabilizing the origin
- output tracking

despite the presence of the disturbance.

$\rightsquigarrow$ First we have to discuss *finite-time stability*.

Section 1

Finite-Time Stability

Finite-Time Stability

Consider ($f : \mathbb{R}^n \to \mathbb{R}^n$)

$$\dot{x} = f(x), \qquad x(0) = x_0 \in \mathbb{R}^n$$

We assume $f(0) = 0$, and thus $x^e = 0$ is an equilibrium.

Definition (Finite-time stability)

The origin is said to be finite-time stable if there exist an open neighborhood $\mathcal{D} \subset \mathbb{R}^n$ of the origin and a function $T : \mathcal{D}\backslash\{0\} \to (0, \infty)$, called the settling-time function, such that the following statements hold:

- (Stability) For every $\varepsilon > 0$ there exists a $\delta > 0$ such that, for every $x(0) = x_0 \in \mathcal{B}_\delta \cap \mathcal{D}\backslash\{0\}$, $x(t) \in \mathcal{B}_\varepsilon$ for all $t \in [0, T(x_0))$.
- (Finite-time convergence) For every $x(0) = x_0 \in \mathcal{D}\backslash\{0\}$, $x(\cdot)$ is defined on $[0, T(x_0))$, $x(t) \in \mathcal{D}\backslash\{0\}$ for all $t \in [0, T(x_0))$, and $x(t) \to 0$ for $t \to T(x_0)$.

The origin is said to be a globally finite-time stable if it is finite-time stable with $\mathcal{D} = \mathbb{R}^n$.

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Example

Consider

$$\dot{x} = f(x) = -\sqrt[3]{x^2}, \qquad \text{(with } f(0) = 0\text{)}$$

Note that

- f is not Lipschitz at the origin
- uniqueness of solutions can only be guaranteed if $x(t) \neq 0$

We can verify that

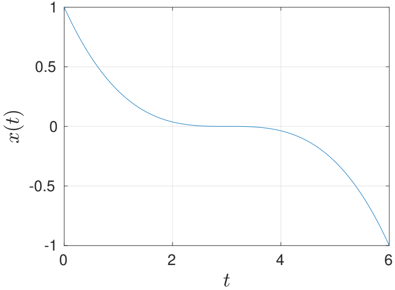
$$x(t) = -\tfrac{1}{27}\left(t - 3\,\mathrm{sign}(x(0))\sqrt[3]{|x(0)|}\right)^3$$

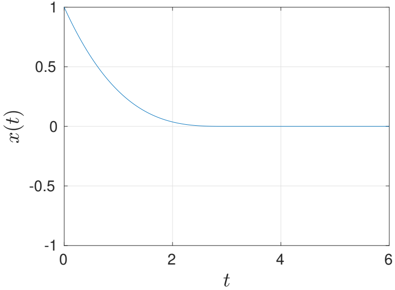
is a solution for all $x \in \mathbb{R}$.
However, for $x(0) > 0$

$$x(t) = \begin{cases} -\frac{1}{27}\left(t - 3\sqrt[3]{|x(0)|}\right)^3 & \text{if } t \leq 3\sqrt[3]{|x(0)|} \\ 0 & \text{if } t \geq 3\sqrt[3]{|x(0)|} \end{cases}$$

is also a solution.

Finite-Time Stability (2)





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Finite-Time Stability (3)

Example

Consider

$$\dot{x} = f(x) = -\operatorname{sign}(x)\sqrt[3]{x^2}.$$

We can verify

$$x(t) = \begin{cases} -\frac{1}{27}\operatorname{sign}(x(0))(t - 3\sqrt[3]{|x(0)|})^3 & \text{if } t \leq 3\sqrt[3]{|x(0)|} \\ 0 & \text{if } t \geq 3\sqrt[3]{|x(0)|} \end{cases}$$

$\rightsquigarrow$ The ODE admits unique solutions

Once the equilibrium is reached, the inequalities

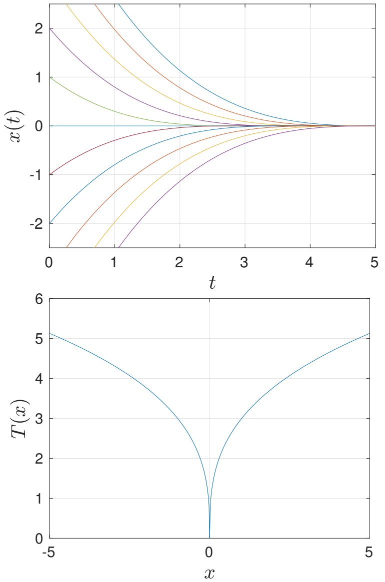
$$-\operatorname{sign}(x)\sqrt[3]{x^2} < 0 \text{ for all } x > 0, \quad \text{and}$$

$$-\operatorname{sign}(x)\sqrt[3]{x^2} > 0 \text{ for all } x < 0$$

ensure that the origin is attractive.

It follows from the explicit solution that

- The origin is finite-time stable
- Settling time $T(x) = 3\sqrt[3]{|x|}$



Finite-Time Stability (4)

Theorem (Lyapunov fcn for finite-time stability)

Consider $\dot{x} = f(x)$ *with* $f(0) = 0$. *Assume there exist a continuous function* $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, *which is continuously differentiable on* $\mathbb{R}^n \backslash \{0\}$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ *and a constant* $\kappa > 0$ *such that*

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|),$$
$$\dot{V}(x) = \langle \nabla V(x), f(x) \rangle \leq -\kappa\sqrt{V(x)} \qquad \forall x \neq 0.$$

Then the origin is globally finite-time stable.
Moreover, the settling-time $T(x) : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *is upper bounded by*

$$T(x) \leq \frac{2}{\kappa}\sqrt{\alpha_2(|x|)}.$$

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Proof.

Comparison principle:

$$\int_0^t \frac{dV(x(t))}{\sqrt{V(x(t))}} \leq -\int_0^t \kappa \, dt$$

leads to

$$\sqrt{V(x(t))} \leq \sqrt{V(x(0))} - \frac{\kappa t}{2}.$$

Using the lower and upper bound

$$|x(t)| \leq \alpha_1^{-1} \left(\left(\sqrt{\alpha_2(|x(0)|)} - \frac{\kappa t}{2} \right)^2 \right)$$

$\rightsquigarrow$ Finite-time convergence
Moreover,

$$\sqrt{\alpha_2(|x(0)|)} - \frac{\kappa t}{2} = 0,$$

implies $|x(T)| \leq 0$ for all $T \geq \frac{2}{\kappa} \sqrt{\alpha_2(|x(0)|)}$ □

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Moreover, the settling-time $T(x) : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is upper bounded by

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For quadratic Lyapunov functions: $(a_1, a_2 > 0)$

$$a_1|x|^2 \leq V(x) \leq a_2|x|^2$$
$$|x(t)| \leq \tfrac{1}{\sqrt{a_1}} \left(\sqrt{a_2}|x(0)| - \tfrac{\kappa t}{2} \right)$$
$$T(|x|) \leq |x| \frac{2\sqrt{a_2}}{\kappa}$$

Proof.

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Finite-Time Stability (5)

Example

Consider again

$$\dot{x} = f(x) = -\operatorname{sign}(x)\sqrt[3]{x^2}$$

Candidate Lyapunov function (continuously differentiable for all $x \neq 0$)

$$V(x) = \sqrt[3]{x^2}$$

Define $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$,

$$\alpha_1(s) = \alpha_2(s) = \sqrt[3]{x^2}$$

Then, for all $x \neq 0$, it holds that

$$\begin{aligned}\dot{V}(x) &= \langle \nabla V(x), -\operatorname{sign}(x)\sqrt[3]{x^2}\rangle = \tfrac{2}{3}\operatorname{sign}(x)|x|^{-\frac{1}{3}}(-\operatorname{sign}(x)|x|^{\frac{2}{3}}) \\ &= -\tfrac{2}{3}|x|^{\frac{1}{3}} = -\tfrac{2}{3}\sqrt{V(x)}\end{aligned}$$

$\rightsquigarrow$ V is a Lyapunov function & the origin is finite-time stable
Bound on the settling time

$$T(x) \leq \tfrac{2}{\kappa}\sqrt{\alpha_2(|x|)} = \frac{2}{\frac{2}{3}}\sqrt{|x|^{\frac{2}{3}}} = 3\sqrt[3]{|x|}$$

Section 2

Basic Sliding Mode Control

Basic Sliding Mode Control

As an example, consider:

$$\dot{x} = x^3 + z,$$
$$\dot{z} = u + \delta(t, x, z).$$

- Unknown disturbance $\delta : \mathbb{R}_{\geq 0} \times \mathbb{R}^2 \to \mathbb{R}$
- Assumption: there exists $L_\delta \in \mathbb{R}_{>0}$ such that

$$|\delta(t, x, z)| \leq L_\delta \qquad (t, x, z) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^2$$

- Thus, δ is bounded but not necessarily continuous

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Goal: Exponential stability of the x-subsystem

- I.e., we want x to behave as $\dot{x} = -x$ (for all bounded disturbances)
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Approach: Define a new state

$$\sigma \doteq x^3 + z + x \quad \text{and} \quad V(\sigma) = \tfrac{1}{2}\sigma^2$$

- Then

$$\begin{aligned} \dot{V}(\sigma) &= \sigma\dot{\sigma} = \sigma\left(3x^2\dot{x} + \dot{z} + \dot{x}\right) \\ &= \sigma\left(3x^5 + 3x^2 z + u + \delta(t, x, z) + x^3 + z\right). \end{aligned}$$

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- Selecting $v = -\rho\ \mathrm{sign}(\sigma)$, $\rho > 0$, provides the estimate

$$\dot{V}(\sigma) = \sigma\left(-\rho\ \mathrm{sign}(\sigma) + \delta(t, x, z)\right) = -\rho|\sigma| + \sigma\delta(t, x, z)$$
$$\leq -\rho|\sigma| + L_\delta|\sigma| = -(\rho - L_\delta)|\sigma|.$$

- Finally, with $\rho = L_\delta + \frac{\kappa}{\sqrt{2}}$, $\kappa > 0$, we have

$$\dot{V}(\sigma) \leq -\frac{\kappa|\sigma|}{\sqrt{2}} = -\alpha\sqrt{V(\sigma)}$$ $\rightsquigarrow$ finite-time stab. of $\sigma = 0$

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- Note that the control

$$u = -\left(L_\delta + \tfrac{\kappa}{\sqrt{2}}\right)\mathrm{sign}\left(x^3 + z + x\right) - 3x^5 - 3x^2 z - x^3 - z$$

is independent of the term $\delta(t, x, z)$.

Basic Sliding Mode Control (2)

Consider:

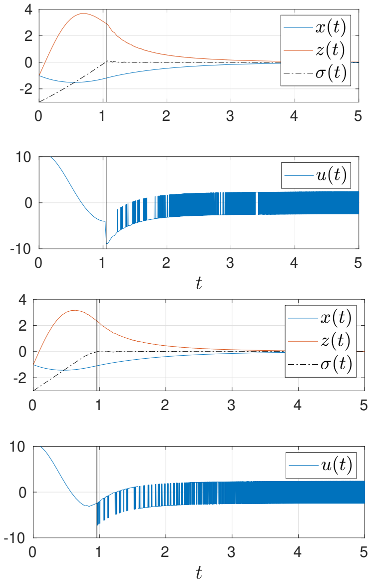
$$\dot{x} = x^3 + z,$$
$$\dot{z} = u + \delta(t, x, z).$$

Control law:

$$u = -\left(L_\delta + \tfrac{\kappa}{\sqrt{2}}\right) \operatorname{sign}\left(x^3 + z + x\right) - 3x^5 - 3x^2 z - x^3 - z$$

Parameter selection for the simulations:

- $L_\delta = 1$ and $\kappa = 2$
- $\delta(t, x, z) = \sin(t)$ (top)
- $\delta(t, x, z) = \operatorname{sign}(\cos(2t)\sin(2t))$ (bottom)



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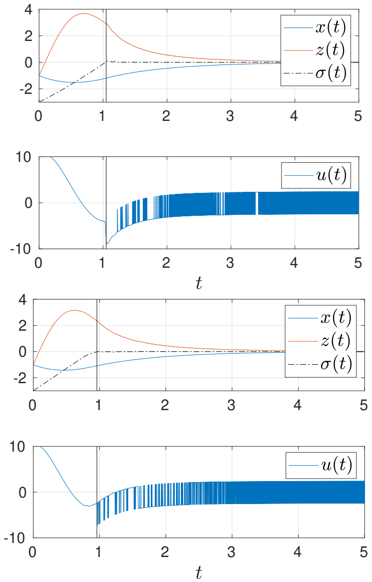
We observe that

- σ converges to zero in finite-time
- Afterwards (x, z) asymptotically approach the origin
- Since the ordinary differential equation is solved numerically, σ is not exactly zero!



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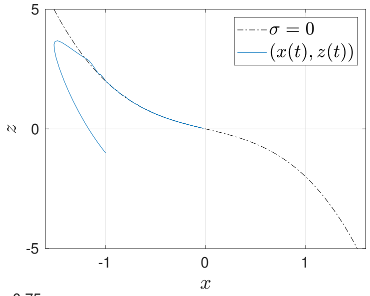
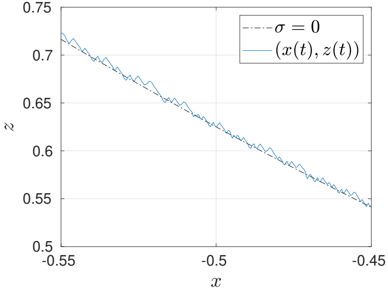
Convergence structure:

⇝ Similar to backstepping/forwarding



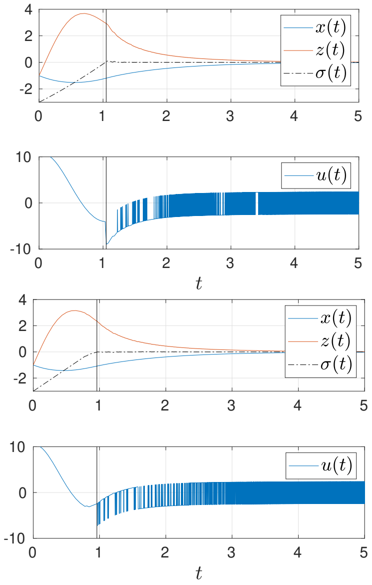


Terminology

- Sliding variable: σ
- Sliding surface

$$\{(x,z) \in \mathbb{R}^2 : \sigma(x,z) = 0,\ (x,z) \in \mathbb{R}^2\},$$

⇝ The sliding variable, and thus implicitly the sliding surface, is defined such that the origin of the x-subsystem is exponentially stable if $\sigma(t) = 0$, $t \in \mathbb{R}_{\geq 0}$, is satisfied.



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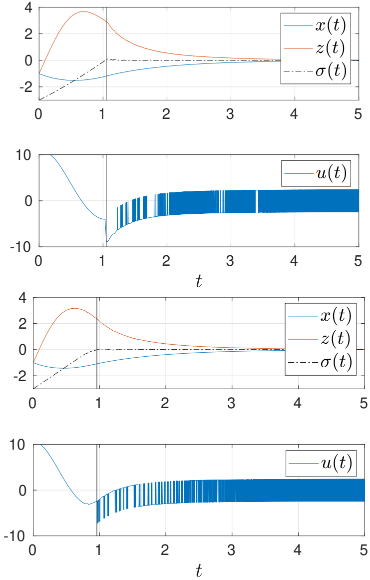
- We have defined σ based on the condition $\dot{x} = -x$. We could have also used $\dot{x} = -2x$ or $\dot{x} = -x^3$ (asymptotic stability), for example.
- The control law u is derived such that states converge to the sliding surface in finite-time.



Terminology

- Sliding variable: σ
- Sliding surface

$$\{(x,z) \in \mathbb{R}^2 : \sigma(x,z) = 0,\ (x,z) \in \mathbb{R}^2\},$$

⇝ The sliding variable, and thus implicitly the sliding surface, is defined such that the origin of the x-subsystem is exponentially stable if $\sigma(t) = 0$, $t \in \mathbb{R}_{\geq 0}$, is satisfied.

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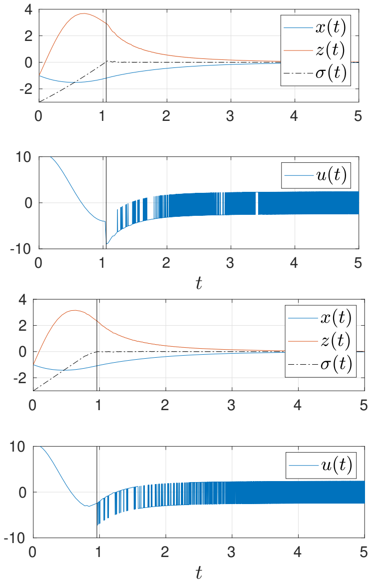
⇝ Convergence of $\sigma(t) \to 0$ is called the *reaching phase*.

- On the sliding surface the selection of u ensures that the dynamics behave like $\dot{x} = -x$.

⇝ This is called *sliding phase* and guarantees asymptotic stability of the origin for the overall closed-loop system.



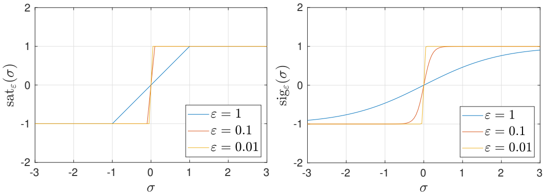
Chattering & Chattering Avoidance

Note that:

- The control law u is discontinuous due to $v = \rho \operatorname{sign}(\sigma)$
- v switches between ρ and $-\rho$ depending on the sign of the sliding variable σ

$\rightsquigarrow$ Chattering (because σ is always slightly off)

Chattering & Chattering Avoidance



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Approximation of the sign-function:

- (Continuous) saturation: $(\mathrm{sat}_\varepsilon : \mathbb{R} \to [-1, 1])$

$$\mathrm{sat}_\varepsilon(\sigma) = \mathrm{sat}(\tfrac{x}{\varepsilon}) = \begin{cases} 1, & \frac{\sigma}{\varepsilon} \geq 1 \\ \frac{\sigma}{\varepsilon}, & -1 \leq \frac{\sigma}{\varepsilon} \leq 1 \\ -1, & \frac{\sigma}{\varepsilon} \leq -1 \end{cases}$$

- (Smooth) *Sigmoid function*: $(\mathrm{sig}_\varepsilon : \mathbb{R} \to [-1, 1])$

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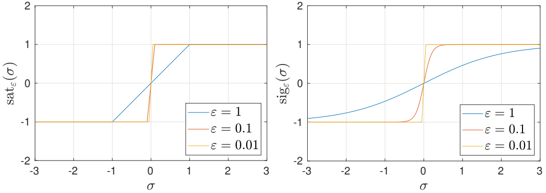

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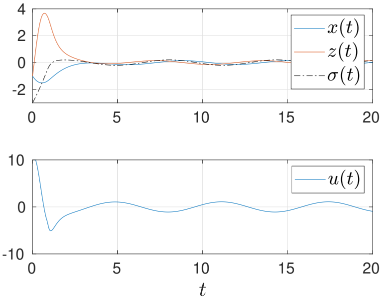
Back to the example: $(\varepsilon = 0.5)$

$$u_\varepsilon = -(L_\delta + \tfrac{\kappa}{\sqrt{2}})\,\mathrm{sat}_\varepsilon(x^3 + z + x) - 3x^5 - 3x^2 z - x^3 - z$$



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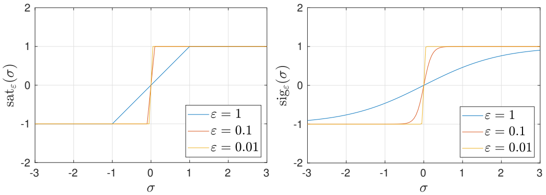

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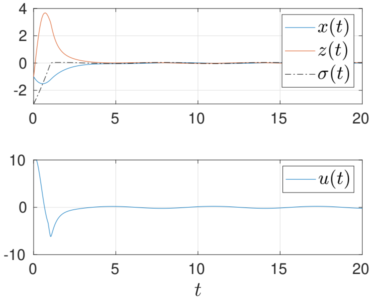

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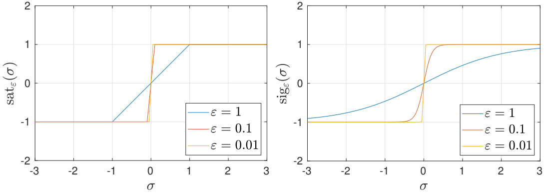

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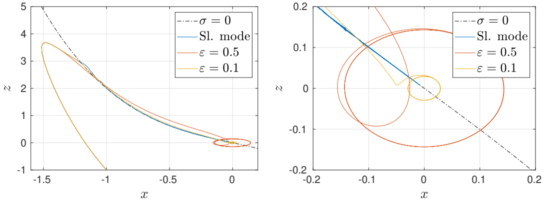

Section 3

A More General Structure

A More General Structure

Consider:

$$\dot{x} = f_1(x, z)$$
$$\dot{z} = f_2(x, z) + g(x, z)(u + \delta(t, x, z))$$

where

- $f_1 : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, $f_1(0, 0) = 0$
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Assumptions:

- $|\delta(t, x, z)| \leq L_\delta$ for all $(t, x, z) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^{n+1}$
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- Design virtual control law $z = k(x)$ such that $x = 0$ for $\dot{x} = f_1(x, k(x))$ is asymptotically stable
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$$\dot{\sigma} = \dot{z} - \tfrac{dk}{dt}(x)$$
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- Consider candidate Lyapunov function: $V(\sigma) = \frac{1}{2}\sigma^2$:

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- Select the new input:

$$v = -\left(\tfrac{\kappa}{\sqrt{2}} + L_g L_\delta\right)\operatorname{sign}(\sigma)$$

- Thus, with the bounds it holds that:

$$\dot{V}(\sigma) \leq \sigma v + |\sigma| L_g L_\delta = -\tfrac{\kappa}{\sqrt{2}}|\sigma| = -\kappa\sqrt{V(\sigma)}$$

$\rightsquigarrow$ $\sigma(t) = 0$ in finite time.

A More General Structure (2)

Theorem

Consider the dynamics

$$\begin{aligned}\dot{x} &= f_1(x,z)\\ \dot{z} &= f_2(x,z) + g(x,z)(u + \delta(t,x,z))\end{aligned}$$

with $f_1(0,0) = 0$. Assume that $g(x,z) \neq 0$ and there exists a constant $L_g > 0$ such that $|g(x,z)| \leq L_g$ for all $(x,z) \in \mathbb{R}^{n+1}$. Additionally assume that $k(x)$ is defined such that the origin of $\dot{x} = f(x,k(x))$ is asymptotically stable with $k(0) = 0$.
Then for all disturbances δ satisfying the condition

$$|\delta(t,x,z)| \leq L_\delta \qquad \forall\ (t,x,z) \in \mathbb{R}^{n+1} \times \mathbb{R}_{\geq 0}$$

for some $L_\delta > 0$, the feedback law

$$u = \frac{1}{g(x,z)}\left(-f_2(x,z) + \frac{\partial k}{\partial x}(x) f_1(x,z)\right) - \left(\frac{\kappa}{\sqrt{2}} + L_g L_\delta\right)\frac{\text{sign}(z - k(x))}{g(x,z)}$$

asymptotically stabilizes the origin of the system for all $\kappa > 0$.
Additionally, the sliding surface $\sigma = z - k(x) = 0$ is reached no later than

$$T(\sigma(0)) = T(z(0) - k(x(0))) = \tfrac{1}{\sqrt{2}\kappa}|z(0) - k(x(0))|.$$

Section 4

Estimating the Disturbance

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So far:

- We have introduced the sliding variable σ
- Through Lyapunov arguments we ensure that (in theory) $\sigma(t) = 0$ in finite time
- In numerical simulations (or in practice) σ will not be exactly zero.
- To dominate the disturbance, we defined the control law

$$u = \frac{\left(-f_2(x,z) + \frac{\partial k}{\partial x}(x) f_1(x,z)\right) - \left(\frac{\kappa}{\sqrt{2}} + L_g L_\delta\right) \operatorname{sign}(z - k(x))}{g(x,z)}$$

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Now:

- Consider the **unimplementable control**

$$u_{eq} = \frac{1}{g(x,z)} \left(-f_2(x,z) + \frac{\partial k}{\partial x}(x) f_1(x,z)\right) - \delta(t,x,z)$$

(which is called the *equivalent control*)

- Note that: (Assuming that δ is sufficiently smooth,) on the sliding surface where $\dot{\sigma} = 0$, it follows that u_{eq} guarantees $\sigma(t) = 0$ for all $t \geq T$ if $\sigma(T) = 0$ without the chattering effects
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Estimation of δ through a low-pass filter:

- Idea: Apply low-pass filter to the chattering input

$$v = -\left(\frac{\kappa}{\sqrt{2}} + L_g L_\delta\right) \operatorname{sign}(\sigma).$$

- In particular, consider augmented dynamics ($\tau > 0$ small)

$$\begin{aligned}
\dot{x} &= f_1(x,z) \\
\dot{z} &= f_2(x,z) + g(x,z)(\hat{u}_{eq} + \delta(t,x,z)) \\
\dot{\xi} &= -\tfrac{1}{\tau}\xi + \tfrac{1}{\tau} \operatorname{sign}(z - k(x))
\end{aligned}$$

- Approximated equivalent control:

$$\begin{aligned}
\hat{u}_{eq} = &\frac{\left(-f_2(x,z) + \frac{\partial k}{\partial x}(x) f_1(x,z)\right)}{g(x,z)} \\
&- \left(\frac{\kappa}{\sqrt{2}} + L_g L_\delta\right) \frac{\xi}{g(x,z)}.
\end{aligned}$$

(where we have replaced $\operatorname{sign}(\sigma)$ by ξ)

$\rightsquigarrow$ The approximated equivalent control is an alternative to u

Estimating the Disturbance (Detour: Low-pass filter)

Remark (Low-pass filter)

The dynamics $\dot{\xi} = -\frac{1}{\tau}\xi + \frac{1}{\tau}\operatorname{sign}(z - k(x))$ represent a *low-pass filter*. To see this, consider the one-dimensional system

$$\begin{aligned} \dot{x} &= -\tfrac{1}{\tau}x + \tfrac{1}{\tau}u \\ y &= x \end{aligned}$$

and its representation in the frequency domain

$$\hat{y}(s) = (s + \tfrac{1}{\tau})^{-1}\tfrac{1}{\tau}\hat{u}(s) = \frac{\frac{1}{\tau}}{s + \frac{1}{\tau}}\hat{u}(s).$$

For $\tau > 0$ small we observe from the transfer function

$$G(s) = \frac{\frac{1}{\tau}}{s + \frac{1}{\tau}}$$

that for low frequencies the system approximately satisfies $\hat{y}(s) \approx \hat{u}(s)$ and for high frequencies it holds that $\hat{y}(s) \approx 0$.

Estimating the Disturbance (Example)

Example

Original example with augmented state: ($\tau > 0$, small)

$$\dot{x} = x^3 + z,$$
$$\dot{z} = u + \delta(t, x, z)$$
$$\dot{\xi} = -\tfrac{1}{\tau}\xi + \tfrac{1}{\tau}\,\mathrm{sign}(z - k(x))$$

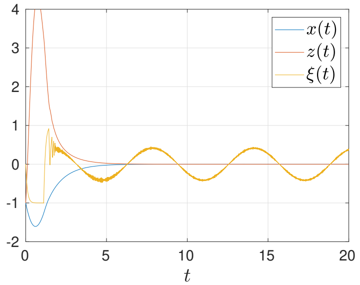
- We follow the steps so far, define $z = k(x) = -x^3 - x$, i.e., $\dot{x} = x^3 - x^3 - x = -x$ (i.e. $x = 0$ is exponentially stable).
- The sliding mode control law (from the theorem)

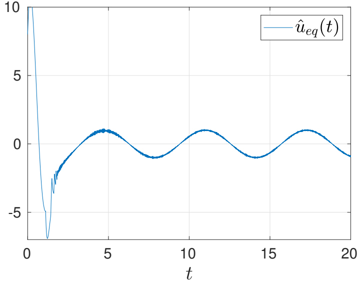
$$u = (-3x^2 - 1)(x^3 + z) - \left(\tfrac{\kappa}{\sqrt{2}} + L_\delta\right)\mathrm{sign}(z + (x^3 + x))$$
$$= -3x^5 - x^3 - 3x^2 z - z - \left(\tfrac{\kappa}{\sqrt{2}} + L_\delta\right)\mathrm{sign}(z + x^3 + x)$$

- The approximated equivalent control

$$\hat{u}_{eq} = -3x^5 - x^3 - 3x^2 z - z - \left(\tfrac{\kappa}{\sqrt{2}} + L_\delta\right)\xi$$

Here: $\delta(t, x, z) = \sin(t)$, $L_\delta = 1$, $\kappa = 2$ and $\tau = 0.1$.





Estimating the Disturbance

Compare equivalent & approximated equivalent control:

$$u_{eq} = \frac{1}{g(x,z)}\left(-f_2(x,z) + \frac{\partial k}{\partial x}(x) f_1(x,z)\right) - \delta(t,x,z)$$

$$\hat{u}_{eq} = \frac{\left(-f_2(x,z)+\frac{\partial k}{\partial x}(x) f_1(x,z)\right)}{g(x,z)} - \left(\frac{\kappa}{\sqrt{2}} + L_g L_\delta\right)\frac{\xi}{g(x,z)}$$

An estimation of the disturbance:

$$\hat{\delta}(t,x,z) = \left(\frac{\kappa}{\sqrt{2}} + L_g L_\delta\right)\frac{\xi}{g(x,z)}$$

Estimating the Disturbance

Compare equivalent & approximated equivalent control:

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An estimation of the disturbance:

$$\hat{\delta}(t,x,z) = \left(\frac{\kappa}{\sqrt{2}} + L_g L_\delta\right)\frac{\xi}{g(x,z)}$$

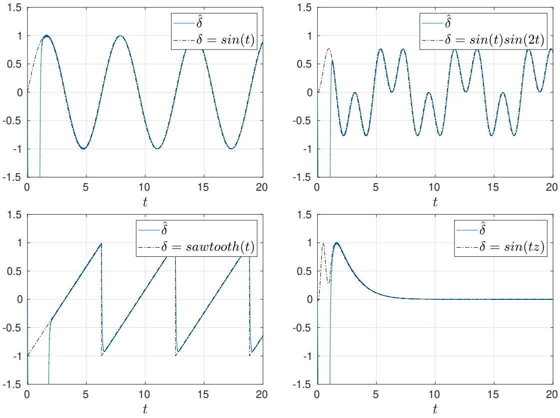
Example

Consider (u sliding mode control law)

$$\dot{x} = x^3 + z,$$
$$\dot{z} = u + \delta(t,x,z)$$
$$\dot{\xi} = -\frac{1}{0.05}\xi + \frac{1}{0.05}\operatorname{sign}(z - k(x))$$

Estimated disturbance:

$$\hat{\delta}(t,x,z) = (\sqrt{2}+1)\xi$$



Section 5

Output Tracking

Output Tracking

So far:

- (Asymptotic) Stabilization of the origin

Now:

- Tracking of a reference signal

Output Tracking

So far:

- (Asymptotic) Stabilization of the origin

Now:

- Tracking of a reference signal

In particular, consider

- $y = x, \qquad \dot{x} = x^3 + z, \qquad \dot{z} = u + \delta(t, x, z)$
- reference signal $y_r : \mathbb{R}_{\geq 0} \to \mathbb{R}$ twice cont. diff.

Goal:

- $y(t) \to y_r(t)$ for $t \to \infty$

Output Tracking

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- (Asymptotic) Stabilization of the origin

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Goal:

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Controller design:

- Define error dynamics

$$e(t) = y_r(t) - y(t) \quad \text{and demand} \quad e(t) \overset{t\to\infty}{\to} 0$$

- and the requirement that $e(t) \to 0$ for $t \to \infty$.
- Define the sliding variable (based on $\dot{e} = -e$)

$$\sigma = \dot{e} + e = \dot{y}_r - \dot{y} + y_r - y = \dot{y}_r + y_r - x^3 - z - x$$

- Calculating the time derivative:

$$\dot{\sigma} = \ddot{y}_r + \dot{y}_r - 3x^2\dot{x} - \dot{z} - \dot{x}$$

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- Calculating the time derivative:

$$\dot{\sigma} = \ddot{y}_r + \dot{y}_r - 3x^2\dot{x} - \dot{z} - \dot{x}$$

- Candidate Lyapunov function $V(\sigma) = \frac{1}{2}\sigma^2$:

$$\dot{V}(\sigma) = \dot{\sigma}\sigma$$
$$= \sigma(\ddot{y}_r + \dot{y}_r - 3x^5 - 3x^2 z - u - \delta(t, x, z) - x^3 - z)$$

- Define the input (with new degree of freedom v):

$$u = -3x^5 - 3x^2 z - x^3 - z - v$$

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$$\begin{aligned}\dot{V}(\sigma) &= \dot{\sigma}\sigma \\ &= \sigma(\ddot{y}_r + \dot{y}_r - 3x^5 - 3x^2 z - u - \delta(t,x,z) - x^3 - z)\end{aligned}$$

- Define the input (with new degree of freedom v):

$$u = -3x^5 - 3x^2 z - x^3 - z - v$$

- Assume we do not know the reference $y_r(t)$ to be tracked *a priori*, define the 'new disturbance'

$$\psi(t, x, z) = \ddot{y}_r + \dot{y}_r - \delta(t, x, z)$$

Output Tracking

So far:

- (Asymptotic) Stabilization of the origin

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In particular, consider

- $y = x, \qquad \dot{x} = x^3 + z, \qquad \dot{z} = u + \delta(t, x, z)$
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Controller design:

- Define error dynamics

$$e(t) = y_r(t) - y(t) \quad \text{and demand} \quad e(t) \overset{t\to\infty}{\to} 0$$

- and the requirement that $e(t) \to 0$ for $t \to \infty$.
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$$\dot{V}(\sigma) = \dot{\sigma}\sigma$$
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- Define the input (with new degree of freedom v):

$$u = -3x^5 - 3x^2 z - x^3 - z - v$$

- Assume we do not know the reference $y_r(t)$ to be tracked *a priori*, define the 'new disturbance'

$$\psi(t, x, z) = \ddot{y}_r + \dot{y}_r - \delta(t, x, z)$$

- Then, the candidate Lyapunov functions satisfies

$$\dot{V}(\sigma) = \sigma(\psi(t, x, z) + v)$$

- Assume that $|\psi(t, x, z)| \leq L_\psi$, for $L_\psi > 0$ & define

$$v = -(L_\psi + \tfrac{\kappa}{\sqrt{2}})\,\text{sign}(\sigma), \qquad (\kappa > 0)$$

- Then $\dot{V}(\sigma)$ satisfies

$$\dot{V}(\sigma) = \sigma(\psi(t, x, z) - (L_\psi + \tfrac{\kappa}{\sqrt{2}})\,\text{sign}(\sigma))$$
$$\leq |\sigma| L_\psi - |\sigma| \left(L_\psi + \tfrac{\kappa}{\sqrt{2}}\right) = -\kappa\sqrt{V(x)}$$

Output Tracking

Example

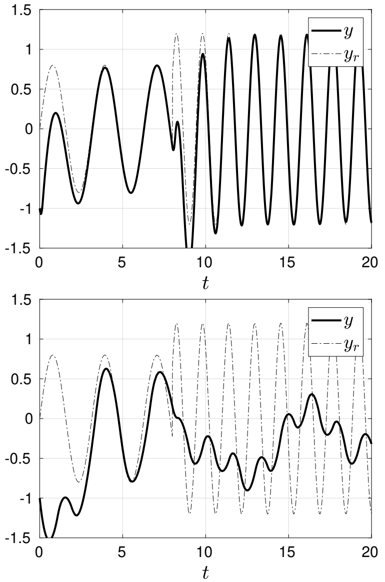
Consider the system

$$y = x, \qquad \dot{x} = x^3 + z, \qquad \dot{z} = u + \delta(t, x, z)$$

with output together with the reference signal

$$y_r(t) = \begin{cases} 0.8\sin(2t) & \text{for } t < 8, \\ 1.2\sin(4t) & \text{for } t \geq 8. \end{cases}$$

- y_r is twice continuously differentiable for all $t \neq 8$
- For the simulation, the disturbance $\delta(t, x, z) = \sin(t)$ is used
- The new disturbance $\psi = \ddot{y}_r + \dot{y}_r - \delta$ satisfies $|\psi(t, x, z)| \leq 25$ for all $t \neq 8$.
- Top figure: $L_\psi = 25$
- Bottom figure: $L_\psi = 1$
- Additionally: $\kappa = 2$



Introduction to Nonlinear Control

Stability, control design, and estimation

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Part III:

Chapter 10: Sliding Mode Control

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