Introduction to Nonlinear Control

Stability, control design, and estimation

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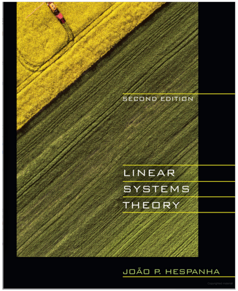
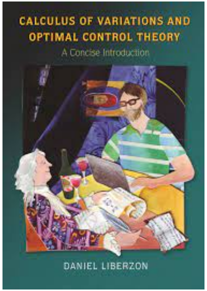
Part II:

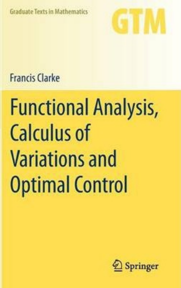
Chapter 14: Optimal Control

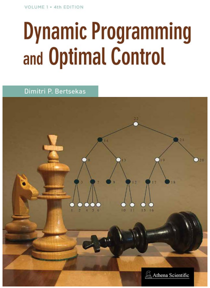
Australian National University

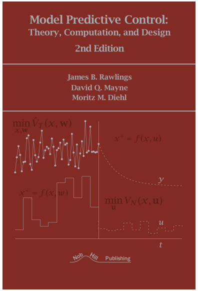
Optimal Control

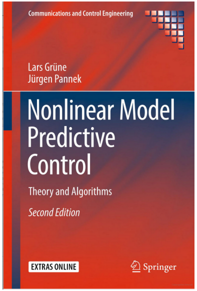












Optimal Control

Section 1

Optimal Control – Continuous Time Setting

Optimal Control – Continuous Time Setting

We consider continuous time system

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0 \in \mathbb{R}^n \qquad (1)$$

By assumption

- $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ locally Lipschitz continuous

Set of inputs and set of solutions:

$$\mathcal{U} = \{u(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}^m |\ u(\cdot) \text{ measurable}\}$$
$$\mathcal{X} = \{x(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}^n |\ x(\cdot) \text{ is absolutely continuous}\}$$

We say that

- $(x(\cdot), u(\cdot)) \in \mathcal{X} \times \mathcal{U}$ is a *solution pair* if it satisfies (1) for almost all $t \in \mathbb{R}_{\geq 0}$.

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Note that:

- The condition *for almost all* $t \in \mathbb{R}_{\geq 0}$ allows a larger class of solutions $x(\cdot)$.
- It is sufficient if $x(\cdot)$ is continuously differentiable for almost all $t \geq 0$.
- $u(\cdot)$ can be piecewise continuous, for example.
- If the initial condition is important (or not clear from context), we use $x(\cdot; x_0) \in \mathcal{X}$ and $u(\cdot; x_0) \in \mathcal{U}$

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For $(x(\cdot), u(\cdot)) \in \mathcal{X} \times \mathcal{U}$ we define

- *Cost functional* (or performance criterion) $J : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R} \cup \{\pm\infty\}$ as

$$J(x_0, u(\cdot)) = \int_0^\infty \ell(x(\tau), u(\tau))d\tau.$$

- *Running cost*: $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$
- *(Optimal) Value function*: $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$,

$$V(x_0) = \min_{u(\cdot) \in \mathcal{U}} J(x_0, u(\cdot))$$

(We assume that the minimum exists!)

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- Sometimes, we use the notation

$$V(x_0) = \min_{u(\cdot) \in \mathcal{U}} J(x_0, u(\cdot)) \quad \text{subject to (1)}.$$

Note that: x_0, and $u(\cdot)$ are sufficient to describe $x(\cdot)$
Optimization in terms of $u(\cdot)$:

$$u^\star(\cdot) = \arg \min_{u(\cdot) \in \mathcal{U}} J(x_0, u(\cdot)).$$

Optimal Control – Continuous Time Setting (2)

Keep in mind:

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We hope to find a feedback law: ($\mu : \mathbb{R}^n \to \mathbb{R}^m$)

$$\mu(x^\star(t)) = u^\star(t) \qquad \forall\, t \in \mathbb{R}_{\geq 0}.$$

Here

- $(x^\star(\cdot), u^\star(\cdot)) \in \mathcal{X} \times \mathcal{U}$ is an optimal solution pair
- $x^\star(\cdot)$ uniquely defined through $u^\star(\cdot)$ and $x^\star(0) = x_0$

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Definition (Feedback invariant)

Consider $\dot{x}(t) = f(x(t), u(t))$. $H : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$ is called feedback invariant with respect to $\mathcal{X} \times \mathcal{U}$ if for all solution pairs $(x_1(\cdot), u_1(\cdot)), (x_2(\cdot), u_2(\cdot)) \in \mathcal{X} \times \mathcal{U}$ with $x_1(0) = x_2(0)$ the equality

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Thus, note that:

- The value of a feedback invariant $H(x(\cdot), u(\cdot))$ depends only on x_0 (and is independent of $u(\cdot)$)

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- Let $H(x(\cdot), u(\cdot))$ be a feedback invariant. If there exists $\Lambda : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ such that

$$J(x_0, u(\cdot)) = H(x(\cdot), u(\cdot)) + \int_0^\infty \Lambda(x(\tau), u(\tau))d\tau$$

where $\quad \min_{u\in\mathbb{R}^m} \Lambda(x, u) = 0, \qquad \forall\, x \in \mathbb{R}^n$

then

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Moreover, it holds that

$$\begin{aligned} V(x_0) &= \min_{u(\cdot)\in\mathcal{U}} J(x_0, u(\cdot)) \\ &= \min_{u(\cdot)\in\mathcal{U}} \left(H(x(\cdot), u(\cdot)) + \textstyle\int_0^\infty \Lambda(x(\tau), u(\tau))\, d\tau\right) \\ &= H(x(\cdot), u(\cdot)) + \textstyle\int_0^\infty \min_{u(\cdot)\in\mathcal{U}} \left(\Lambda(x(\tau), u(\tau))\right)\, d\tau \\ &= H(x(\cdot), u(\cdot)) \end{aligned}$$

Linear Quadratic Regulator

Consider ($A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$)

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \in \mathbb{R}^n$$

To ensure that $H(x(\cdot), u(\cdot)) < \infty$ we define

$$\mathcal{X}_s = \left\{x(\cdot) \in \mathcal{X} : \lim_{t\to\infty} x(t) = 0\right\}$$

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Theorem (Feedback invariant)

Consider the linear system with solution pairs $(x(\cdot), u(\cdot)) \in \mathcal{X}_s \times \mathcal{U}$. Then, for any $P \in \mathcal{S}^n$, the functional $H : \mathcal{X}_s \times \mathcal{U}$ defined as

$$H(x(\cdot), u(\cdot)) = -\int_0^\infty (Ax(\tau) + Bu(\tau))^T Px(\tau) + x^T(\tau)P(Ax(\tau) + Bu(\tau))\, d\tau$$

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Proof.

Consider $(x(\cdot), u(\cdot)) \in \mathcal{X}_s \times \mathcal{U}$. Then

$$-\int_0^\infty \frac{d}{d\tau}\left(x^T(\tau)Px(\tau)\right) d\tau = -x^T(\tau)Px(\tau)\Big|_0^\infty$$
$$= x^T(0)Px(0) - \lim_{t\to\infty} x^T(t)Px(t) = x^T(0)Px(0).$$

Since $x(\cdot) \in \mathcal{X}_s$ by assumption, $x^T(t)Px(t) \overset{t\to\infty}{\to} 0$ vanishes and the term depends only on $x(0)$. □

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We continue with the cost functional: ($Q \in \mathcal{S}^n_{\geq 0}$, $R \in \mathcal{S}^m_{>0}$)

$$J(x_0, u(\cdot)) = \int_0^\infty \left(x^T(\tau)Qx(\tau) + u^T(\tau)Ru(\tau)\right) d\tau$$

($\rightsquigarrow$ Linear quadratic regulator (LQR))

Linear Quadratic Regulator (2)

We add and subtract the feedback invariant:

$$J(x_0,u(\cdot)) = H(x(\cdot),u(\cdot)) + \int_0^\infty x^T Q x + u^T R u + (Ax+Bu)^T P x + x^T P (Ax+Bu)\, d\tau.$$

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Rearranging terms, completing the squares and note that R^{-1} is well defined (since $R > 0$):

$$\begin{aligned}\int_0^\infty x^T Qx + u^T Ru + (Ax+Bu)^T Px + x^T P(Ax+Bu)\; d\tau &= \int_0^\infty x^T(Q + A^T P + PA)x + u^T Ru + 2u^T B^T Px\, d\tau \\ &= \int_0^\infty \left(x^T(Q + A^T P + PA)x + u^T Ru + 2u^T B^T Px + x^T PB^T R^{-1} BPx - x^T PB^T R^{-1} BPx\right) d\tau \\ &= \int_0^\infty \left(x^T(Q + A^T P + PA - PBR^{-1}B^T P)x + (u + R^{-1}B^T Px)^T R(u + R^{-1}B^T Px)\right) d\tau.\end{aligned}$$

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If P can be chosen so that $A^T P + PA + Q - PBR^{-1}B^T P = 0$ cost function reduces to

$$J(x_0, u(\cdot)) = H(x(\cdot), u(\cdot)) + \int_0^\infty (u + R^{-1}B^T P x)^T R (u + R^{-1}B^T P x) d\tau.$$

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Since $R > 0$,

$$\Lambda(x,u) \doteq (u + R^{-1}B^T P x)^T R (u + R^{-1}B^T P x)$$

has a minimum at zero given by $\mu(x(t)) = u(t) = -R^{-1}B^T P x(t)$.

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We add and subtract the feedback invariant:

$$J(x_0,u(\cdot)) = H(x(\cdot),u(\cdot)) + \int_0^\infty x^T Qx + u^T Ru + (Ax+Bu)^T Px + x^T P(Ax+Bu)\, d\tau.$$

Rearranging terms, completing the squares and note that R^{-1} is well defined (since $R > 0$):

$$\int_0^\infty x^T Qx + u^T Ru + (Ax+Bu)^T Px + x^T P(Ax+Bu)\; d\tau = \int_0^\infty x^T(Q + A^T P + PA)x + u^T Ru + 2u^T B^T Px \, d\tau$$

$$= \int_0^\infty \left(x^T(Q + A^T P + PA)x + u^T Ru + 2u^T B^T Px + x^T PB^T R^{-1} BPx - x^T PB^T R^{-1} BPx \right) d\tau$$

$$= \int_0^\infty \left(x^T(Q + A^T P + PA - PBR^{-1}B^T P)x + (u + R^{-1}B^T Px)^T R(u + R^{-1}B^T Px) \right) d\tau.$$

If P can be chosen so that $A^T P + PA + Q - PBR^{-1}B^T P = 0$ cost function reduces to

$$J(x_0,u(\cdot)) = H(x(\cdot),u(\cdot)) + \int_0^\infty (u + R^{-1}B^T Px)^T R(u + R^{-1}B^T Px) d\tau.$$

Since $R > 0$,

$$\Lambda(x,u) \doteq (u + R^{-1}B^T Px)^T R(u + R^{-1}B^T Px)$$

has a minimum at zero given by $\mu(x(t)) = u(t) = -R^{-1}B^T Px(t)$.

- For $H(x(\cdot),u(\cdot)) = -\int_0^\infty (Ax+Bu)^T Px + x^T P(Ax+Bu)\; d\tau$ to be a feedback invariant $\lim_{t\to\infty} x(t) = 0$ needs to be satisfied (i.e., $(x(\cdot),u(\cdot)) \in \mathcal{X}_s \times \mathcal{U}$) thus $A - BR^{-1}B^T P$ needs to be Hurwitz.

Linear Quadratic Regulator (3)

Linear system:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \in \mathbb{R}^n$$

Quadratic cost function:

$$J(x_0, u(\cdot)) = \int_0^\infty \left(x^T(\tau) Q x(\tau) + u^T(\tau) R u(\tau) \right) d\tau$$

Theorem

Consider the linear system and the quadratic cost function defined through $Q \in \mathcal{S}^n_{\geq 0}$, $R \in \mathcal{S}^m_{>0}$. If there exists $P \in \mathcal{S}^n$ satisfying the continuous time algebraic Riccati equation

$$A^T P + PA + Q - PBR^{-1}B^T P = 0$$

and if $A - BR^{-1}B^T P$ is Hurwitz, then $\mu(x) = -R^{-1}B^T Px$ minimizes the quadratic cost function and the optimal value function is given by

$$V(x_0) = x_0^T P x_0.$$

Linear Quadratic Regulator (4)

Theorem (Linear quadratic regulator)

Consider the linear system with output $y(t) = Cx(t)$, $y \in \mathbb{R}^p$, *and assume that* (A, B) *is stabilizable and* (A, C) *is detectable. Let* $Q \in \mathcal{S}^p$, $R \in \mathcal{S}^m$ *and* $S \in \mathbb{R}^{p \times m}$ *be such that*

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} > 0$$

and consider the quadratic cost function

$$J(x_0, u(\cdot)) = \int_0^\infty \begin{bmatrix} x(\tau)^T C^T & u(\tau)^T \end{bmatrix} \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} Cx(\tau) \\ u(\tau) \end{bmatrix} d\tau.$$

Then the following properties are satisfied.

1. *The Riccati equation*

$$A^T P + PA + C^T QC - (PB + C^T S)R^{-1}(B^T P + S^T C) = 0$$

   has a unique positive definite solution $P \in \mathcal{S}^n_{>0}$.

2. *The state feedback* $\mu(x) = -R^{-1}(B^T P + S^T C)x$ *ensures that the closed loop matrix* $A - BR^{-1}(B^T P + S^T C)$ *is Hurwitz.*

3. *The optimal value function minimizing the cost function is given by* $V(x_0) = x_0^T P x_0$ *and* V *is a Lyapunov function of the closed loop system.*

Control-Affine Nonlinear Systems

Control-affine nonlinear systems (with equilibrium $x = 0$):

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t), \qquad x(0) = x_0 \in \mathbb{R}^n,$$

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Theorem

Consider the control affine nonlinear system with solution pairs $(x(\cdot), u(\cdot)) \in \mathcal{X}_s \times \mathcal{U}$. Then for a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$,

$$H(x(\cdot), u(\cdot)) = -\int_0^\infty \big(L_f V(x(\tau)) + L_g V(x(\tau))u(\tau)\big)\, d\tau$$

is a feedback invariant.

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Consider

- $R : \mathbb{R}^n \to \mathbb{R}^{m \times m}$, $R(x)$ positive definite and bounded away from zero for all $x \in \mathbb{R}^n$, i.e., there exists a $c > 0$ such that $R(x) - cI > 0$ for all $x \in \mathbb{R}^n$.
- $Q : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ positive definite
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As before we can write

$$J(x(\cdot), u(\cdot)) = H(x(\cdot), u(\cdot))$$
$$+ \int_0^\infty Q(x) + L_f V(x) - \tfrac{1}{4} L_g V(x)(R(x))^{-1} L_g V(x)^T$$
$$+ (u + \tfrac{1}{2}(R(x))^{-1} L_g V(x))^T R(x)(u + \tfrac{1}{2}(R(x))^{-1} L_g V(x)) d\tau$$

Theorem

Consider the control-affine system and the cost function. If there exists a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ such that for all $x \in \mathbb{R}^n$

$$Q(x) + L_f V(x) - \tfrac{1}{4} L_g V(x)(R(x))^{-1} L_g V(x)^T = 0,$$

and if the feedback

$$\mu(x) = -\tfrac{1}{2}(R(x))^{-1} L_g V(x)$$

asymptotically stabilizes the origin, then this feedback minimizes $J(x(\cdot), u(\cdot))$.

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$\rightsquigarrow$ Checking asymptotic stability is not straightforward.
$\rightsquigarrow$ If V is a CLF, then asymptotic stability follows.

Inverse Optimality

So far, we followed the standard approach of optimal control, i.e.,

- the designer specifies a cost function to be minimized

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Now, consider the reverse process, i.e.,

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$$\mu(x) = -\tfrac{1}{2}(R(x))^{-1} L_g V(x)$$

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$\rightsquigarrow$ Compute

$$Q(x) = -L_fV(x) + \tfrac{1}{4}L_gV(x)(R(x))^{-1}L_gV(x)^T$$

(Since V is a CLF, Q is positive definite.

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(Since V is a CLF, Q is positive definite.

⇝ The control law $\mu(x)$ minimizes

$$J(x_0, u(\cdot)) = \int_0^\infty \Big(Q(x(\tau)) + u^T(\tau) R(x(\tau)) u(\tau) \Big)\, d\tau$$

with the computed functions Q and R.

⇝ $\mu(x)$ is referred to as *inverse optimal*

- In particular, not the cost function is specified by the designer, but rather the stabilizing feedback which defined the cost function

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Recall the ISS redesign & Sontag's formula:

- The concept of inverse optimality allows an analysis of the control laws obtained through the ISS redesign and Sontag's universal formula by calculating the performance criterion for which the controllers are optimal.)

Section 2

Optimal Control – Discrete Time Setting

Optimal Control – Discrete Time Setting

Consider

$$x(k+1) = f(x(k), u(k)), \qquad x(0) = x_0 \qquad (2)$$

- Set of inputs and set of solutions:

$$\mathcal{U} = \{u(\cdot) : \mathbb{N}_0 \to \mathbb{R}^m\}, \qquad \mathcal{X} = \{x(\cdot) : \mathbb{N}_0 \to \mathbb{R}^n\}.$$

$$\mathcal{X}_s = \left\{x(\cdot) \in \mathcal{X} : \lim_{k\to\infty} x(k) = 0\right\}.$$

- Cost functional

$$J(x_0, u(\cdot)) = \sum_{k=0}^{\infty} \ell(x(k), u(k)).$$

(with running costs $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$)

- Optimal value function:

$$V(x_0) = \min_{u(\cdot)\in\mathcal{U}} J(x_0, u(\cdot))$$

$$\text{subject to (2)}$$

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$$V(x_0) = \min_{u(\cdot)\in\mathcal{U}} J(x_0, u(\cdot))$$

subject to (2)

Definition (Feedback invariant)

Consider $x^+ = f(x, u)$. $H : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$ is called feedback invariant with respect to $\mathcal{X} \times \mathcal{U}$ if for all solution pairs $(x_1(\cdot), u_1(\cdot)), (x_2(\cdot), u_2(\cdot)) \in \mathcal{X} \times \mathcal{U}$ with $x_1(0) = x_2(0)$ the equality

$$H(x_1(\cdot), u_1(\cdot)) = H(x_2(\cdot), u_2(\cdot)) \qquad \text{holds.}$$

- Decomposition of the cost function

$$J(x_0, u(\cdot)) = H(x(\cdot), u(\cdot)) + \sum_{k=0}^{\infty} \Lambda(x(k), u(k))$$

with

$$\min_{u\in\mathbb{R}^m} \Lambda(x, u) = 0 \qquad \forall\, x \in \mathbb{R}^n$$

$\rightsquigarrow$ Optimal feedback stabilizer

$$\mu(x(k)) = \arg\min_{u\in\mathbb{R}^m} \Lambda(x(k), u).$$

The Linear Quadratic Regulator

Consider the linear system

$$x(k+1) = Ax(k) + Bu(k), \quad x(0) = x_0 \in \mathbb{R}^n$$

Theorem

Consider the discrete time linear system with solution pairs $(x(\cdot), u(\cdot)) \in \mathcal{X}_s \times \mathcal{U}$. Then, for any symmetric matrix $P \in \mathcal{S}^n$, the functional $H : \mathcal{X}_s \times \mathcal{U} \to \mathbb{R}$ defined as

$$H(x(\cdot), u(\cdot))$$

$$= -\sum_{k=0}^{\infty} (Ax(k)+Bu(k))^T P(Ax(k)+Bu(k)) - x(k)^T Px(k)$$

is a feedback invariant.

($\leadsto$ Note the structure of the discrete time Lyapunov equation)

Theorem

Consider the discrete time linear system and the quadratic cost function

$$J(x_0, u(\cdot)) = \sum_{k=0}^{\infty} x(k)^T Qx(k) + u(k)^T Ru(k)$$

defined through $Q \in \mathcal{S}^n_{\geq 0}$, $R \in \mathcal{S}^m_{>0}$. If there exists $P \in \mathcal{S}^n$ satisfying the discrete time algebraic Riccati equation

$$Q + A^T PA - P - A^T PB \left(R + B^T PB\right)^{-1} B^T PA = 0$$

and if

$$A - B(R + B^T PB)^{-1} B^T PA$$

is a Schur matrix, then

$$\mu(x) = -\left(R + B^T PB\right)^{-1} B^T PAx$$

minimizes the cost function and the optimal value function is given by $V(x_0) = x_0^T Px_0$.

The Linear Quadratic Regulator (2)

Theorem

Consider the discrete time linear system and the quadratic cost function

$$J(x_0, u(\cdot)) = \sum_{k=0}^{\infty} x(k)^T Qx(k) + u(k)^T Ru(k)$$

defined through $Q \in \mathcal{S}^n_{\geq 0}$, $R \in \mathcal{S}^m_{>0}$. *If there exists* $P \in \mathcal{S}^n$ *satisfying the discrete time algebraic Riccati equation*

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minimizes the cost function and the optimal value function is given by $V(x_0) = x_0^T Px_0$.

Proof.

Same steps as in the continuous time setting:

$$J(x_0, u(\cdot)) = H(x(\cdot), u(\cdot)) + \sum_{k=0}^{\infty} x(k)^T Qx(k) + u(k)^T Ru(k)$$
$$+ \sum_{k=0}^{\infty} (Ax(k) + Bu(k))^T P (Ax(k) + Bu(k)) - x(k)^T Px(k)$$

Define $\tilde{R} = R + B^T PB$. Then $J(x_0, u(\cdot))$ can be rewritten

$$\sum_{k=0}^{\infty} x^T Qx + u^T Ru + (Ax + Bu)^T P (Ax + Bu) - x^T Px$$
$$= \sum_{k=0}^{\infty} x^T (Q + A^T PA - P)x + u^T \tilde{R}u + 2u^T B^T PAx$$
$$+ \sum_{k=0}^{\infty} x^T A^T PB\tilde{R}^{-1} B^T PAx - x^T A^T PB\tilde{R}^{-1} B^T PAx$$
$$= \sum_{k=0}^{\infty} x^T (Q + A^T PA - P - A^T PB\tilde{R}^{-1} B^T PA)x$$
$$+ \sum_{k=0}^{\infty} (\tilde{R}u + B^T PAx)^T \tilde{R}^{-1} (\tilde{R}u + B^T PAx).$$

- P positive definite $\rightsquigarrow$ $B^T PB$ positive semidefinite $\rightsquigarrow$ $\tilde{R} = R + B^T PB$ positive definite $\rightsquigarrow$ $\tilde{R}^{-1}$ well defined
- We recover the algebraic Riccati equation
- We recover the feedback law

□

The Linear Quadratic Regulator (3)

Theorem (The discrete time linear quadratic regulator)

Consider the linear system with output $y(k) = Cx(k)$, $y \in \mathbb{R}^p$, *and assume that the pair* (A, B) *is stabilizable and* (A, C) *is detectable. Let* $Q \in \mathcal{S}^p$, $R \in \mathcal{S}^m$ *and* $S \in \mathbb{R}^{p\times m}$ *be such that*

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} > 0$$

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$$J(x_0, u(\cdot)) = \sum_{k=0}^{\infty} \begin{bmatrix} x(k)^T C^T & u(k)^T \end{bmatrix} \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} Cx(k) \\ u(k) \end{bmatrix}.$$

1. *The Riccati equation*

   $$C^T QC + A^T PA - P - (A^T PB + C^T S)\left(R + B^T PB\right)^{-1}(B^T PA + S^T C) = 0$$

   has a unique positive definite solution $P \in \mathcal{S}^n_{>0}$.
2. *The state feedback* $\mu(x) = -\left(R + B^T PB\right)^{-1}(B^T PA + S^T C)x$ *ensures that the closed loop matrix* $A - B\left(R + B^T PB\right)^{-1}(B^T PA + S^T C)$ *is a Schur matrix.*
3. *The optimal value function minimizing the cost function is given by* $V(x_0) = x_0^T P x_0$ *and* V *defines a Lyapunov function of the closed loop system.*

Section 3

From Infinite to Finite Dimensional Optimization

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$$x(k+1) = f(x(k), u(k)), \qquad x(0) = x_0 \qquad (3)$$

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(Optimal control problem)

- Optimal solution pair

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- $(x^\star(\cdot), u^\star(\cdot)) \in \mathcal{X} \times \mathcal{U}$ is optimal with respect to a specific measure (i.e., a specific cost functional).
- To obtain the optimal solution pair an infinite dimensional optimization problem needs to be solved.

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$\rightsquigarrow$ In general only possible under specific assumptions on the system dynamics and the input and solution space $\mathcal{U}$ and $\mathcal{X}$.

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⇝ In general only possible under specific assumptions on the system dynamics and the input and solution space $\mathcal{U}$ and $\mathcal{X}$.

⇝ Even if it is possible to solve the optimization problem for a specific initial value $x_0 \in \mathbb{R}^n$, to obtain an optimal feedback law $\mu(x(t))$ (instead of an optimal open loop input $u(t)$) it needs to be solved for all $x_0 \in \mathbb{R}^n$.

From Infinite to Finite Dimensional Optimization

Consider

$$x(k+1) = f(x(k), u(k)), \qquad x(0) = x_0 \qquad (3)$$

- Cost functional

$$J(x_0, u(\cdot)) = \sum_{k=0}^{\infty} \ell(x(k), u(k)).$$

  (with running costs $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$)

- Optimal value function:

$$V(x_0) = \min_{u(\cdot) \in \mathcal{U}} J(x_0, u(\cdot))$$

$$\text{subject to (3)}$$

  (Optimal control problem)

- Optimal solution pair

$$(x^\star(\cdot), u^\star(\cdot)) \in \mathcal{X} \times \mathcal{U}$$

Recapitulation of results:

- $(x^\star(\cdot), u^\star(\cdot)) \in \mathcal{X} \times \mathcal{U}$ is optimal with respect to a specific measure (i.e., a specific cost functional).
- To obtain the optimal solution pair an infinite dimensional optimization problem needs to be solved.

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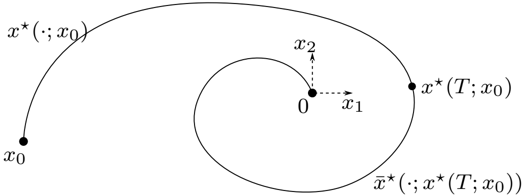
- How can we overcome the restriction to linear dynamics?
- How can we incorporate state/input constraints?
- How can we simplify the infinite horizon (or infinite dimensional) optimization problem?

From Infinite to Finite Dimensional Optimization: The Principle of Optimality

The principle of optimality:

- In words, for any point on an optimal solution $x^\star(\cdot)$, the remaining control inputs $u^\star(\cdot)$ are also optimal.



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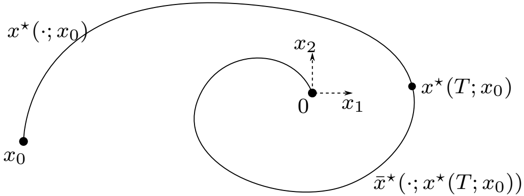
- Assume that solutions of the optimal control problem are unique.



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- For $x_0 \in \mathbb{R}^n$ let $(x^\star(\cdot; x_0), u^\star(\cdot; x_0))$ be the optimal solution pair of

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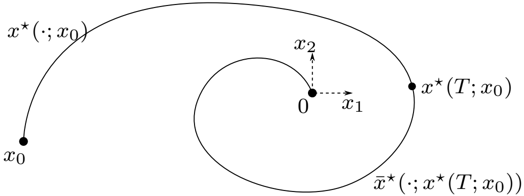

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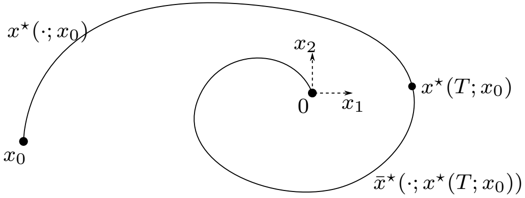

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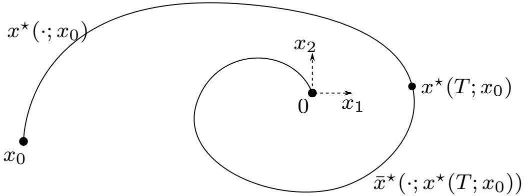

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- Then the principle of optimality states that

$$\bar{u}^\star(\cdot;x^\star(T;x_0)) = u^\star(\cdot + T;x_0)$$
$$\bar{x}^\star(\cdot;x^\star(T;x_0)) = x^\star(\cdot + T;x_0)$$

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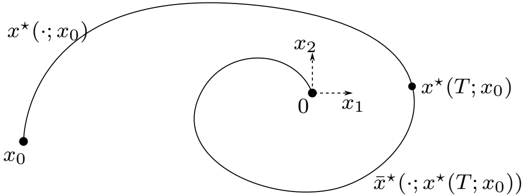

Note that:

- In the case that optimal solutions are not unique, 'only'

$$J(x^\star(T;x), \bar{u}^\star(\cdot;x^\star(T;x))) = J(x^\star(T;x), u^\star(\cdot+T;x))$$

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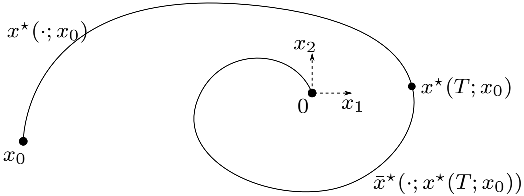

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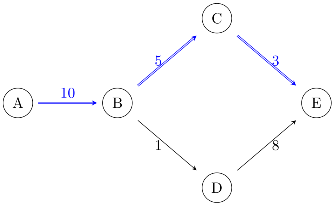
- Same result in the discrete time setting



Constrained Optimal Control for Linear Systems

- Consider

$$x(k+1) = f(x(k), u(k)), \qquad x(0) = x_0$$

- Set of inputs and set of solutions:

$$\mathcal{U} = \{u(\cdot) : \mathbb{N}_0 \to \mathbb{R}^m\}, \qquad \mathcal{X} = \{x(\cdot) : \mathbb{N}_0 \to \mathbb{R}^n\}.$$

- Cost functional

$$\begin{aligned} J(x_0, u(\cdot)) &= \textstyle\sum_{k=0}^{\infty} \ell(x(k), u(k)) \\ &= \textstyle\sum_{k=0}^{\infty} x(k)Qx(k) + u(k)Ru(k) \end{aligned}$$

- Optimal value function:

$$V(x_0) = \min_{u(\cdot)\in\mathcal{U}} J(x_0, u(\cdot))$$

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- We restrict the input space to

$$\mathcal{U}_{\mathbb{U}} = \{u(\cdot) : \mathbb{N}_0 \to \mathbb{R}^m \,|\, u(k) \in \mathbb{U}\ \forall k \in \mathbb{N}\},$$

for $\mathbb{U} \subset \mathbb{R}^m$ closed and convex, $0 \in \mathrm{int}(\mathbb{U})$

- Corresponding OCP:

$$V(x_0) = \min_{u(\cdot)\in\mathcal{U}_{\mathbb{U}}} J(x_0, u(\cdot))$$

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⇝ The approach followed so far is in general not applicable

⇝ Standard (convex) optimization algorithms are not directly applicable since the OCP is infinite dimensional

Constrained Optimal Control for Linear Systems (2)

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- Under the assumption that $x^\star(N) \in \mathbb{X}_F$ it holds that

$$\begin{aligned}\min_{u(\cdot) \in \mathcal{U}_{\mathbb{U}}} J(x_0, u(\cdot)) &= \sum_{k=0}^{\infty} (x^\star)^T Q x^\star + (u^\star)^T R u^\star \\ &= \sum_{k=0}^{N-1} (x^\star(k))^T Q x^\star(k) + (u^\star(k))^T R u^\star(k) \\ &\quad + \sum_{k=N}^{\infty} (x^\star(k))^T Q x^\star(k) + (u^\star(k))^T R u^\star(k) \\ &= \sum_{k=0}^{N-1} (x^\star)^T Q x^\star + (u^\star)^T R u^\star + (x^\star(N))^T P_F x^\star(N)\end{aligned}$$

- Moreover

$$V(x^\star(N)) = (x^\star(N))^T P_F x^\star(N)$$

Constrained Optimal Control for Linear Systems (3)

- Restrict the definitions to a finite horizon:

$$\mathcal{U}^N = \{u_N(\cdot) = (u(0), \ldots, u(N-1)) |\ u(\cdot) \in \mathcal{U}\}$$

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&\min_{u(\cdot)\in\mathcal{U}_{\mathbb{U}}} J(x_0, u(\cdot)) \\
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&= \min_{u_N(\cdot)\in\mathcal{U}_{\mathbb{U}}^N} J_N(x_0, u_N(\cdot)) + x(N)^T P_F x(N)
\end{aligned}$$

$$V(x_0) = \min_{u_N(\cdot)\in\mathcal{U}_{\mathbb{U}}^N} J_N(x_0, u^N(\cdot)) + x(N)^T P_F x(N)$$

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- Alternatively, consider *terminal constraints*

$$\min_{u^N(\cdot)\in\mathcal{U}_{\mathbb{U}}^N} J_N(x_0, u_N(\cdot)) + x(N)^T P_F x(N)$$
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- However
  - in this case the optimal solution might not be optimal with respect to the cost function (i.e., it might be cheaper to reach $\mathbb{X}_F$ in more than N steps)
  - the optimization problem is infeasible if it is not possible to reach the set $\mathbb{X}_F$ in N steps

Dynamic Programming & the Backward Recursion

Now, consider finite horizon OCP (for $N \in \mathbb{N}$ and with $\mathbb{X}_F = \{0\}$):

$$V_N(x_0) = \min_{u_N(\cdot) \in \mathcal{U}_{\mathbb{U}}^N} J_N(x_0, u_N(\cdot))$$

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- Rewriting the optimal value function

$$V_N(x_0) =$$

$$\min_{u(0)\in\mathbb{U}} \ell(x_0, u(0)) + V_{N-1}(f(x_0, u(0)))$$

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$\rightsquigarrow$ If V_{N-1} is known, V_N can be computed by minimizing the OCP with respect to $u(0)$.

$\rightsquigarrow$ If $\mathbb{X}_{\{0\}}^{N-1}$ is known, then $\mathbb{X}_{\{0\}}^N$ can be constructed

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$$\text{subject to dynamics \& initial cond., and } x(N) = 0$$

- The set of states for which the problem is feasible:

$$\mathbb{X}^N_{\{0\}} = \left\{ x_0 \in \mathbb{R}^n \;\middle|\; \exists u^N(\cdot) \in \mathcal{U}^N_{\mathbb{U}} \text{ such that } \begin{array}{l} x(N) = 0 \\ x^+ = f(x,u) \\ x(0) = x_0 \end{array} \right\}$$

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⇝ If V_{N-1} is known, V_N can be computed by minimizing the OCP with respect to $u(0)$.

⇝ If $\mathbb{X}^{N-1}_{\{0\}}$ is known, then $\mathbb{X}^N_{\{0\}}$ can be constructed

- Note that the condition $x(N) \in \{0\}$ can be replaced by alternative conditions on the final state.

Dynamic Programming & the Backward Recursion (Example)

Consider the discrete time system

$$x_1^+ = x_1 + 2x_2, \qquad x_2^+ = -x_2 + 2u$$

together with the running costs $\ell(x,u) = u^2$

- Initialize backward recursion: $V_0(x_0) = 0$ and $\mathbb{X}^0_{\{0\}} = \{0\}$

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- Initialize backward recursion: $V_0(x_0) = 0$ and $\mathbb{X}^0_{\{0\}} = \{0\}$
- $N = 1$: Optimization problem

$$V_1(x_0) = \min_{u(0)\in\mathbb{R}} \quad u(0)^2$$
$$\text{subject to} \quad 0 = x_1(0) + 2x_2(0)$$
$$0 = -x_2(0) + 2u(0)$$

⇝ Rearranging the equality constraints shows that

$$u_1^\star(0) = \tfrac{1}{2}x_2(0) \quad \text{and thus} \quad V_1(x_0) = \tfrac{1}{4}x_2(0)^2.$$

⇝ Feasibility condition: $x_1(0) = -2x_2(0)$, i.e.,

$$\mathbb{X}^1_{\{0\}} = \{x \in \mathbb{R}^2 |\ x_1 = -2x_2\}$$

- $N = 2$: Optimization problem

$$V_2(x_0) = \min_{u(0),u(1)\in\mathbb{R}} \quad u(0)^2 + u(1)^2$$
$$\text{subject to} \quad 0 = -x_1(1) + x_1(0) + 2x_2(0)$$
$$0 = -x_2(1) - x_2(0) + 2u(0)$$
$$0 = x_1(1) + 2x_2(1)$$
$$0 = -x_2(1) + 2u(1).$$

⇝ Using the result from V_1, i.e., $u_1^\star(0) = u(1) = \frac{1}{2}x_2(1)$ and $x_1(1) = -2x_2(1)$, then

$$V_2(x_0) = \min_{u(0)\in\mathbb{R}} \quad u(0)^2 + \tfrac{1}{4}x_2(1)^2$$
$$\text{subject to} \quad 0 = 2x_2(1) + x_1(0) + 2x_2(0)$$
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⇝ Eliminating $x_2(1)$:

$$V_2(x_0) = \min_{u(0)\in\mathbb{R}} \quad u(0)^2 + \tfrac{1}{4}(-x_2(0) + 2u(0))^2$$
$$\text{subject to} \quad 0 = x_1(0) + 4u(0).$$

Dynamic Programming & the Backward Recursion (Example)

- $N = 2$: Optimization problem

$$\begin{aligned} V_2(x_0) = \min_{u(0),u(1)\in\mathbb{R}} \quad & u(0)^2 + u(1)^2 \\ \text{subject to} \quad & 0 = -x_1(1) + x_1(0) + 2x_2(0) \\ & 0 = -x_2(1) - x_2(0) + 2u(0) \\ & 0 = x_1(1) + 2x_2(1) \\ & 0 = -x_2(1) + 2u(1). \end{aligned}$$

⇝ Using the result from V_1, i.e., $u_1^\star(0) = u(1) = \frac{1}{2}x_2(1)$ and $x_1(1) = -2x_2(1)$, then

$$\begin{aligned} V_2(x_0) = \min_{u(0)\in\mathbb{R}} \quad & u(0)^2 + \tfrac{1}{4}x_2(1)^2 \\ \text{subject to} \quad & 0 = 2x_2(1) + x_1(0) + 2x_2(0) \\ & 0 = -x_2(1) - x_2(0) + 2u(0). \end{aligned}$$

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⇝ Thus $u_2^\star(\cdot)$ is given by

$$\begin{aligned} u_2^\star(0) &= -\tfrac{1}{4}x_1(0), \\ u_2^\star(1) &= \tfrac{1}{2}x_2(1) = \tfrac{1}{2}(-x_2(0) + 2u_2^*(0)) \\ &= -\tfrac{1}{2}x_2(0) - \tfrac{1}{4}x_1(0). \end{aligned}$$

⇝ The optimal value function V_2 satisfies

$$\begin{aligned} V_2(x(0)) &= \tfrac{1}{16}x_1(0)^2 + \left(\tfrac{1}{2}x_2(0) + \tfrac{1}{4}x_1(0)\right)^2 \\ &= \tfrac{1}{8}x_1(0)^2 + \tfrac{1}{4}x_1(0)x_2(0) + \tfrac{1}{4}x_2(0)^2 \end{aligned}$$

⇝ The optimization problem is feasible for all $x_0 \in \mathbb{X}^2_{\{0\}} = \mathbb{R}^2$, i.e., the origin can be reached from any point in two discrete time steps.

(This is consistent with the observation that for controllable systems, in general n steps are necessary to reach the origin from an initial state.)

Dynamic Programming & the Backward Recursion (Example, 2)

Consider a general linear system

$$x^+ = Ax + Bu,$$

$x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, together with quadratic running costs

$$\ell(x,u) = x^T Q x + u^T R u, \qquad Q \in \mathcal{S}^n_{\geq 0}, R \in \mathcal{S}^m_{>0}$$

Here,

- we drop the condition that x needs to reach the origin in N steps. Instead we assume that the system needs to be controlled for $N \in \mathbb{N}$ discrete time steps and the behavior of the dynamics $x(k)$ for $k \geq N$ does not matter.
- for $N = 0$ we initialize: $V_0(x_0) = x_0^T P_0 x_0$ for $P_0 = 0 \in \mathcal{S}^n$ and $u^\star(k) = 0$ for all $k \in \mathbb{N}$.

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For $N = 1$:

$$V_1(x_0) = \min_{u^1(\cdot)\in\mathcal{U}^1} x_0^T Qx_0 + u(0)^T Ru(0) + V_0(x^+)$$
$$\text{subject to } x^+ = Ax_0 + Bu(0)$$

which simplifies to

$$V_1(x_0) = \min_{u(0)\in\mathcal{U}^1} x_0^T Qx_0 + u(0)^T Ru(0).$$

⇝ Since $R \in \mathcal{S}^m_{>0}$ the optimal input is given by $u_1^\star(\cdot; x_0) = 0$ and

$$V_1(x_0) = x_0^T P_1 x_0 = x_0^T Q x_0$$

For $N = 2$:

$$V_2(x_0) = \min_{u^2(\cdot)\in\mathcal{U}^2} x_0^T Qx_0 + u(0)^T Ru(0) + V_1(x^+)$$
$$\text{subject to } x^+ = Ax_0 + Bu(0)$$

which can be rewritten as

$$\begin{aligned}
V_2(x_0) &= \min_{u\in\mathbb{R}^m} x_0^T Qx_0 + u^T Ru + (Ax_0 + Bu)^T P_1 (Ax_0 + Bu) \\
&= \min_{u\in\mathbb{R}^m} x_0^T (Q + A^T P_1 A - A^T P_1 B(R + B^T P_1 B)^{-1} B^T P_1 A)x_0 \\
&\quad + ((R + B^T P_1 B)u - B^T P_1 Ax_0)^T (R + B^T P_1 B)^{-1} \\
&\quad \cdot ((R + B^T P_1 B)u - B^T P_1 Ax_0).
\end{aligned}$$

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which simplifies to

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⇝ Since $R \in \mathcal{S}^m_{>0}$ the optimal input is given by $u_1^\star(\cdot; x_0) = 0$ and

$$V_1(x_0) = x_0^T P_1 x_0 = x_0^T Q x_0$$

For $N = 2$:

$$V_2(x_0) = \min_{u^2(\cdot)\in\mathcal{U}^2} x_0^T Q x_0 + u(0)^T R u(0) + V_1(x^+)$$

$$\text{subject to } x^+ = Ax_0 + Bu(0)$$

which can be rewritten as

$$V_2(x_0) = \min_{u\in\mathbb{R}^m} x_0^T Q x_0 + u^T R u + (Ax_0 + Bu)^T P_1 (Ax_0 + Bu)$$

$$= \min_{u\in\mathbb{R}^m} x_0^T (Q + A^T P_1 A - A^T P_1 B (R + B^T P_1 B)^{-1} B^T P_1 A) x_0$$

$$+ ((R + B^T P_1 B)u - B^T P_1 A x_0)^T (R + B^T P_1 B)^{-1}$$

$$\cdot ((R + B^T P_1 B)u - B^T P_1 A x_0).$$

We can thus conclude that

$$u_2^\star(0; x_0) = -(R + B^T P_1 B)^{-1} B^T P_1 A x_0$$

and $V_2(x_0) = x_0^T P_2 x_0$ where

$$P_2 = Q + A^T P_1 A - A^T P_1 B (R + B^T P_1 B)^{-1} B^T P_1 A$$

For $N \in \mathbb{N}$ it holds that

$$u_N^\star(0, x_0) = -(R + B^T P_{N-1} B)^{-1} B^T P_{N-1} A x_0$$

and $V_N(x_0) = x_0^T P_N x_0$ for

$$P_N = Q + A^T P_{N-1} A - A^T P_{N-1} B$$

$$\cdot (R + B^T P_{N-1} B)^{-1} B^T P_{N-1} A$$

⇝ If $P_N = P_{N-1}$ then the algebraic Riccati equation is recovered

Introduction to Nonlinear Control

Stability, control design, and estimation

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Part II:

Chapter 14: Optimal Control

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