Introduction to Nonlinear Control

Stability, control design, and estimation

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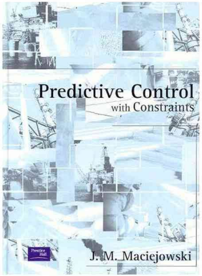
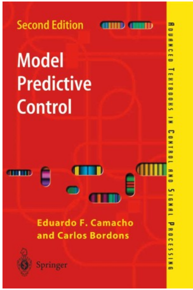
Part II:

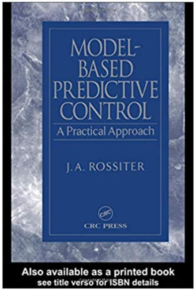
Chapter 15: Model Predictive Control

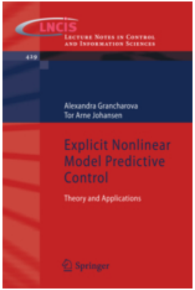
Australian National University

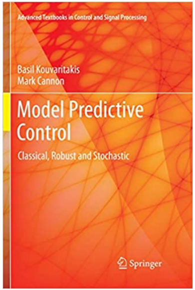
Model Predictive Control

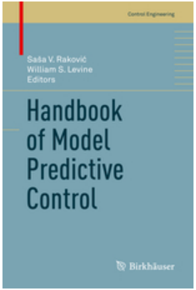


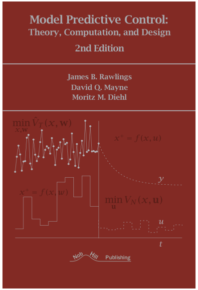


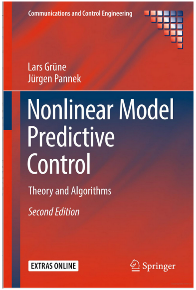








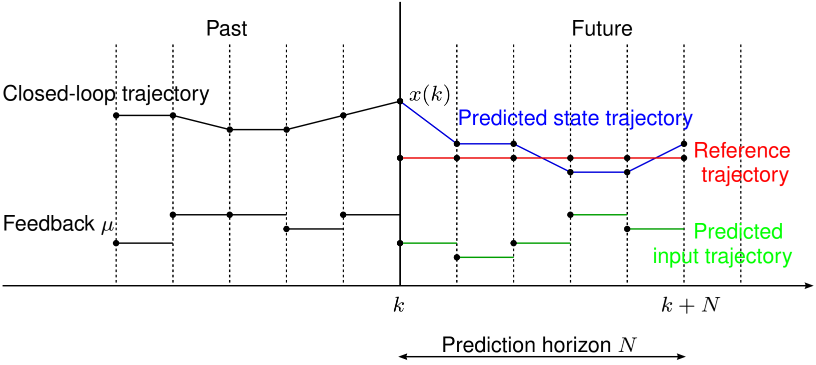




Model Predictive Control

1. MPC Closed-Loop Analysis
   - Performance Estimates
   - Closed Loop Stability Properties
   - Viability & Recursive Feasibility
   - Hard and Soft Constraints
2. Model Predictive Control Schemes
   - Time-Varying Systems & Reference Tracking
   - Linear MPC Versus Nonlinar MPC
   - MPC Without Terminal Costs & Constraints
   - Explicit MPC
   - Economic MPC
3. Implementational Aspects of MPC
   - Warm-Start & Suboptimal MPC
   - Formulation of the Optimization Problem

Model Predictive Control



Here, we consider discrete time systems

$$x^+ = f(x,u), \qquad x(0) = x_0 \in \mathbb{R}^n$$

with $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ $f(0,0) = 0$.

- State constraints $x \in \mathbb{X} \subset \mathbb{R}^n$
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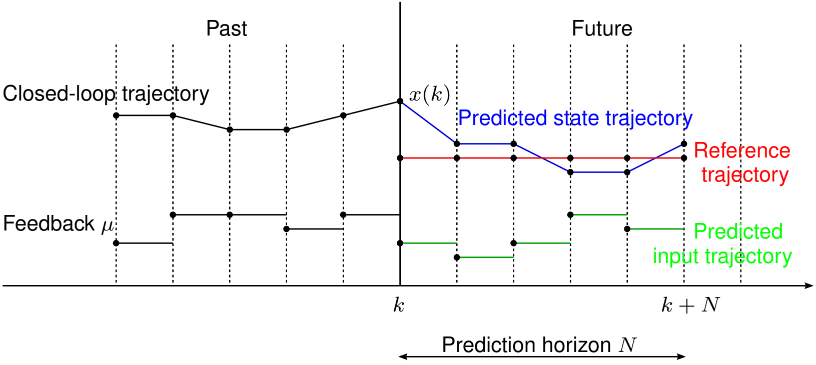

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- If $\mathbb{U}(x)$ is independent of x we write $\mathbb{U} = \mathbb{U}(x)$
- We combine the state and input constraints through

$$\mathbb{D} = \mathbb{X} \times \mathbb{U}(x)$$

- By assumption $(0,0) \in \mathbb{D}$

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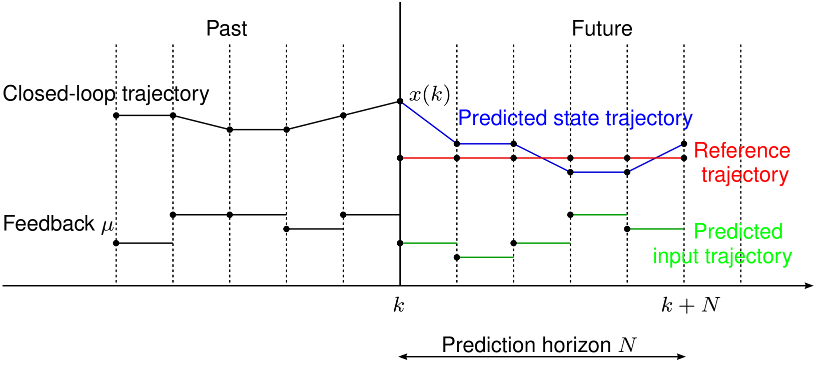

MPC is also known as

- *predictive control*
- *receding horizon control*
- *rolling horizon control*

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Example: Polyhedral Constraints

For $r_1, r_2 \in \mathbb{N}$ consider

$$\Gamma_{x,1} \in \mathbb{R}^{n\times r_1},\ \Gamma_{x,2} \in \mathbb{R}^{n\times r_2},\ \Gamma_u \in \mathbb{R}^{m\times r_2},$$

$\gamma_1 \in \mathbb{R}^{r_1}$ and $\gamma_2 \in \mathbb{R}^{r_2}$.

Then, state constraints can be described through

$$\mathbb{X} = \{x \in \mathbb{R}^n : \Gamma_{x,1} x \leq \gamma_1\}.$$

For a fixed $x \in \mathbb{X}$, we can define the set (i.e., input constraints)

$$\mathbb{U}(x) = \{u \in \mathbb{R}^m : \Gamma_u u \leq \gamma_2 - \Gamma_{x,2} x\}.$$

The state and input constraints:

$$\mathbb{D} = \left\{ (x,u) \in \mathbb{R}^{n+m} \middle| \begin{bmatrix} \Gamma_{x,1} & 0 \\ \Gamma_{x,2} & \Gamma_u \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \leq \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} \right\}$$

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Simple example:

- Consider $\mathbb{X} = [-1,1]^2$ and $\mathbb{U} = [-\frac{1}{4}, \frac{1}{4}]$.

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- Then we can define the matrices and the vectors...

$$\Gamma_{x,1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix},\ \gamma_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

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- ... and combine them

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$$

to obtain a representation for $\mathbb{D}$

The Basic MPC Formulation

- Prediction horizon: $N \in \mathbb{N} \cup \{\infty\}$
- Set of feasible input trajectories of length N (depending on x_0):

$$\mathcal{U}_{\mathbb{D}}^{N} = \left\{ u_N(\cdot) : \mathbb{N}_{[0,N-1]} \to \mathbb{R}^m \;\middle|\; \begin{array}{rcl} x(0) & = & x_0, \\ x(k+1) & = & f(x(k), u(k)), \\ (x(k), u(k)) & \in & \mathbb{D}, \\ & \forall & k \in \mathbb{N}_{[0,N-1]} \end{array} \right\}$$

- We sometimes write $u_N(\cdot; x_0) = u_N(\cdot)$ to highlight the dependence on the initial condition x_0. For clarity, note that

$$u_N(\cdot) = [u_N(0), u_N(1), u(2), \ldots, u_N(N-1)]$$

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$$V_N(x_0) = \min_{u_N(\cdot) \in \mathcal{U}_{\mathbb{D}}^N} J_N(x_0, u_N(\cdot)) + F(x(N))$$

$$\text{subject to dyn. \& init. cond. and } x(N) \in \mathbb{X}_F$$

($\rightsquigarrow$ finite dimensional optimization problem if N is finite)

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- Even if $V_N : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is not known explicitly, for a given $x_0 \in \mathbb{R}^n$, the function $V_N(\cdot)$ can be evaluated in x_0 by solving the OCP.

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- Note that, J_N and V_N are defined as *extended real valued functions* which satisfy $J_N(x_0, u_N(\cdot)) = \infty$ and $V_N(x_0) = \infty$ whenever $\mathcal{U}_{\mathbb{D}}^N = \emptyset$ (i.e., when the OCP is infeasible).
- Here and in the following assume that the minimum in the OCP is attained

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- Here and in the following assume that the minimum in the OCP is attained
- *Optimal open-loop input trajectory* $u_N^\star(\cdot; x_0) \in \mathcal{U}_{\mathbb{D}}^N$ s.t. $x(N) \in \mathbb{X}_F$ &

$$V_N(x_0) = J_N(x_0, u_N^\star(\cdot; x_0)) + F(x(N))$$

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- Even if $V_N : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is not known explicitly, for a given $x_0 \in \mathbb{R}^n$, the function $V_N(\cdot)$ can be evaluated in x_0 by solving the OCP.
- Note that, J_N and V_N are defined as *extended real valued functions* which satisfy $J_N(x_0, u_N(\cdot)) = \infty$ and $V_N(x_0) = \infty$ whenever $\mathcal{U}_{\mathbb{D}}^N = \emptyset$ (i.e., when the OCP is infeasible).
- Here and in the following assume that the minimum in the OCP is attained
- *Optimal open-loop input trajectory* $u_N^\star(\cdot; x_0) \in \mathcal{U}_{\mathbb{D}}^N$ s.t. $x(N) \in \mathbb{X}_F$ &

$$V_N(x_0) = J_N(x_0, u_N^\star(\cdot; x_0)) + F(x(N))$$

- $u_N^\star(\cdot; x_0)$ is used to iteratively define a *feedback law* μ_N, i.e.,

$$\mu_N(x_0) = u_N^\star(0; x_0)$$

$$x_{\mu_N}(k+1) = f(x_{\mu_N}(k), \mu_N(x(k)))$$

The Basic MPC Formulation (2)

Input: Measurement of the initial condition $x(0)$; prediction horizon $N \in \mathbb{N} \cup \{\infty\}$; running cost $\ell : \mathbb{R}^{n+m} \to \mathbb{R}$; constraints $\mathbb{D} \subset \mathbb{R}^{n+m}$; terminal cost $F : \mathbb{R}^n \to \mathbb{R}$ and terminal constraints $\mathbb{X}_F \subset \mathbb{R}^n$.

For $k = 0, 1, 2, \ldots$

1. Measure the current state of the system $x^+ = f(x, u)$ and define $x_0 = x(k)$.
2. Solve the optimal control problem

$$V_N(x_0) = \min_{u_N(\cdot) \in \mathcal{U}_{\mathbb{D}}^N} J_N(x_0, u_N(\cdot)) + F(x(N))$$

$$\text{subject to dyn. \& init. cond. and } x(N) \in \mathbb{X}_F$$

   to obtain the open-loop input $u_N^\star(\cdot; x_0)$.

3. Define the feedback law

$$\mu_N(x(k)) = u_N^\star(0; x_0).$$

4. Compute $x(k+1) = f(x(k), \mu_N(x(k)))$, increment k to $k+1$ and go to 1.

The Basic MPC Formulation (3)

Note that:

- *Optimal open-loop input trajectory*: $u_N^\star(\cdot; x_0)$
- *Optimal open-loop solution* for $k = 0, \ldots, N-2$

$$x_N^\star(0) = x_0$$
$$x_N^\star(k+1) = f(x_N^\star(k), u_N^\star(k; x_0))$$

- In many applications, the discrete time system is an approximation of a plant

$$\dot{x}_p = f_p(x_p, u), \qquad x_p(0) \in \mathbb{R}^n$$

⇝ In this setting the MPC feedback law is usually defined as a *sample-and-hold feedback*

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For $k = 0, 1, 2, \ldots$

1. Measure the current state of the plant $\dot{x}_p = f_p(x_p, u)$ and define $x_0 = x_p(k\Delta)$.
2. Solve the optimal control problem

$$V_N(x_0) = \min_{u_N(\cdot) \in \mathcal{U}_{\mathbb{D}}^N} J_N(x_0, u_N(\cdot)) + F(x(N))$$
$$\text{subject to dyn. \& init. cond. and } x(N) \in \mathbb{X}_F$$

   to obtain the open-loop control law $u_N^\star(\cdot; x_0)$.

3. Define the feedback law

$$\mu_N(x_p(k\Delta)) = u_N^\star(0; x_0).$$

4. Apply the feedback law, i.e., for $t \in [k\Delta, (k+1)\Delta)$ solve

$$\dot{x}_p(t) = f_p(x_p(t), \mu_N(x_p(k\Delta))), \quad x_p(k\Delta) \in \mathbb{R}^n,$$

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Remark

It is not guaranteed that $x_p(\cdot)$ satisfies the state constraints $x_p(t)\in\mathbb{X}$ for all $t\in\mathbb{R}_{\geq 0}$ since the constraints are only enforced at discrete time steps.

Input: Measurement of the initial condition $x_p(0)$; $N\in\mathbb{N}\cup\{\infty\}$; $\ell:\mathbb{R}^{n+m}\to\mathbb{R}$; $\mathbb{D}\subset\mathbb{R}^{n+m}$; $F:\mathbb{R}^n\to\mathbb{R}$ and $\mathbb{X}_F\subset\mathbb{R}^n$; $\Delta>0$.

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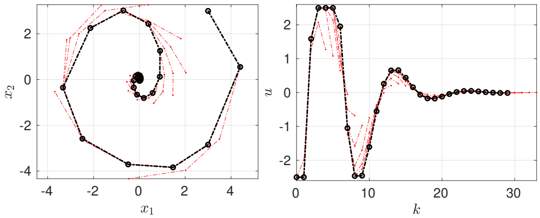
The Basic MPC Formulation (Illustration of properties)



Consider $x^+ = Ax + Bu$ with unstable origin and

$$A = \begin{bmatrix} \frac{6}{5} & \frac{6}{5} \\ -\frac{1}{2} & \frac{6}{5} \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$$

- Prediction horizon: $N = 5$
- The running cost: $\ell(x, u) = x^T x + 5u^2$
- Constraints: $u \in \mathbb{U} = [-2.5, 2.5]$, $x \in \mathbb{R}^2$ (i.e., $\mathbb{D} = \mathbb{R}^2 \times \mathbb{U}$)
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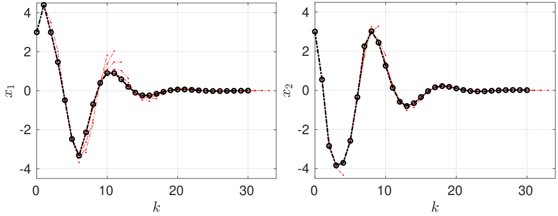

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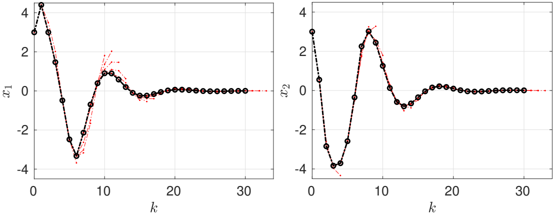
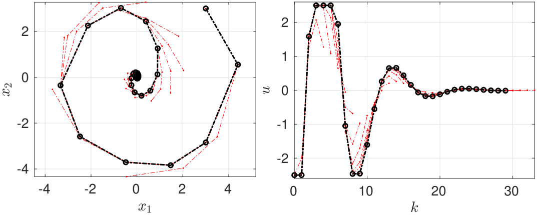
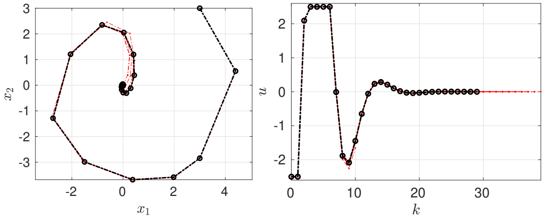




- Now, use the terminal constraint $\mathbb{X}_F = \{0\}$ (which makes $F(x)$ superfluous)
- Prediction horizon $N = 11$ (since for $N < 11$ the OCP is not feasible for $x_0 = [3\ 3]^T$)



The Basic MPC Formulation (Illustration of properties, 2)

Consider again $x^+ = Ax + Bu$ with unstable origin and

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The discrete dynamics define the Euler approximation of

$$\dot{x}_p = A_p x + B_p x = \begin{bmatrix} \frac{1}{5} & \frac{12}{5} \\ -1 & \frac{1}{5} \end{bmatrix} x_p + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u$$

for $\Delta = 0.5$

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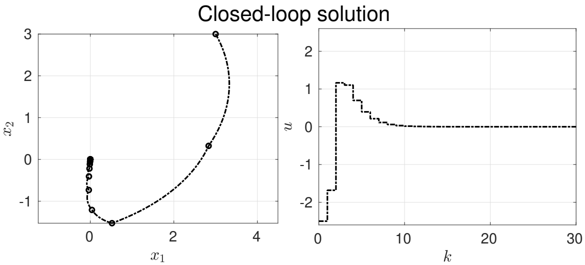

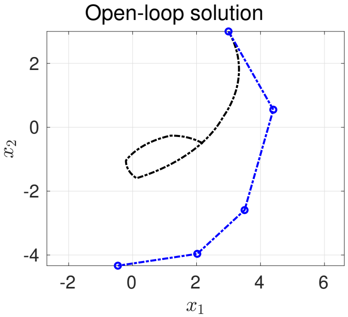

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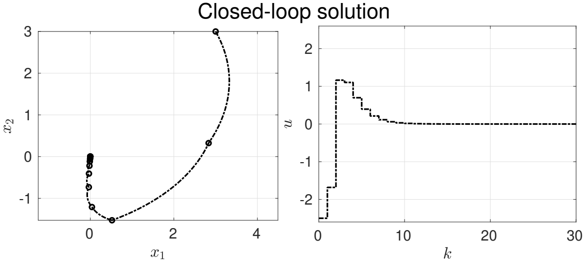

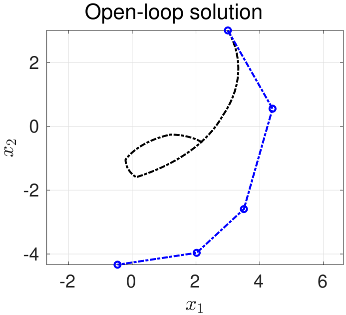

Remark

Since a rather large Δ is used, the two solutions differ significantly. This highlights an important difference between a feedback law and an open loop control law and provides one explanation why in MPC in general only the first piece of $u_N^\star(\cdot)$ is used to define a feedback law.

Section 1

MPC Closed-Loop Analysis

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- MPC can be applied to general nonlinear systems and constraints can be taken into account directly in the controller design.

Disadvantage:

- Since the feedback law is only defined implicitly, the analysis of the closed-loop dynamics is rather difficult.

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- Often the OCP solved in every iteration of the MPC algorithm is a compromise between numerical complexity and optimality.
- In many applications, one is interested in solving the OCP for $N = \infty$.
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- Reasonable questions: What is the relation between
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- Here, the MPC closed-loop costs are defined as

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- It is in general more interesting to establish bounds

$$J_\infty(x_0, \mu_N(\cdot)) \leq \frac{1}{\alpha_N} V_\infty(x_0) \qquad \forall\, x \in \mathbb{R}^n$$

for an $\alpha_N \in (0, 1]$. $\rightsquigarrow$ level of suboptimality

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- For example, if $\alpha_N = \frac{1}{2}$, the MPC closed loop cost is at most twice the infinite horizon optimal control cost.
- Under appropriate assumptions, one can expect $\alpha_N \to 1$ for $N \to \infty$.

$\rightsquigarrow$ Out of the scope of this lecture

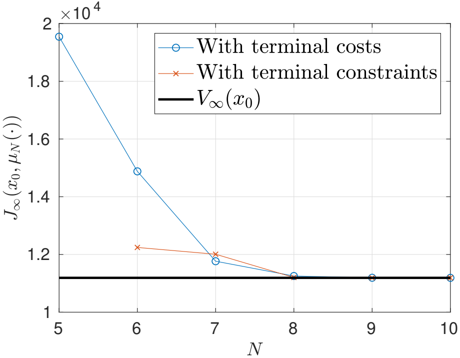
MPC Closed-Loop Analysis (2)

As an example consider:

$$x^+ = Ax + Bu = \begin{bmatrix} 1 & 4 & 0 & 3 & 2 \\ 2 & 4 & 2 & 4 & 2 \\ 3 & 3 & 3 & 0 & 4 \\ 3 & 1 & 3 & 0 & 3 \\ 2 & 3 & 1 & 4 & 4 \end{bmatrix} x + \begin{bmatrix} 2 \\ 3 \\ 1 \\ 2 \\ 3 \end{bmatrix} u$$

- $\ell(x,u) = x^T x + u^2;\quad F(x) = x^T x;\quad \mathbb{X}_F = \{0\};$ $\mathbb{U} = [-40, 40]$

Costs corresponding to $x_0 = [1,1,1,1,1]^T$



(To be precise, $V_\infty(x_0)$ is approximated through $V_{1000}(x_0)$)
Note that:

- The plot only shows the costs for a particular initial condition x_0 and thus, it does not provide an estimate with respect to all initial conditions.
- However, for the particular initial condition, small N lead to almost optimal performance.

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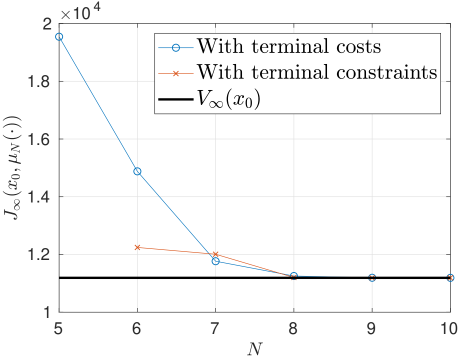
Remark

The performance estimate discussed here compares the MPC closed loop cost with a particular infinite horizon optimal cost functional. To argue that an MPC controller provides nearly optimal performance (if the parameter α_N is close to 1) while operating a plant is only true with respect to the particular infinite horizon cost functional. Thus, the selection of the running cost needs to be well justified when talking about optimality of a controller.

Costs corresponding to $x_0 = [1,1,1,1,1]^T$



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Closed Loop Stability Properties

Consider:

$$x^+ = f(x, \mu_N(x))$$

A standard control application of MPC:

- Stabilization of an equilibrium pair $(x^e, u^e) \in \mathbb{X} \times \mathbb{U}$
- Reasonable running costs: ($Q \geq 0$, $R \geq 0$):

$$\ell(x, u) = (x - x^e)^T Q(x - x^e) + (u - u^e)^T R(u - u^e)$$

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- However: Here, we have assumed (or need to assume) that the optimization problem is feasible for all initial values $x_0 \in \mathbb{X}$!

Closed Loop Stability Properties (Example)

As an example consider:

$$x^+ = Ax + Bu = \begin{bmatrix} 1 & 4 & 0 & 3 & 2 \\ 2 & 4 & 2 & 4 & 2 \\ 3 & 3 & 3 & 0 & 4 \\ 3 & 1 & 3 & 0 & 3 \\ 2 & 3 & 1 & 4 & 4 \end{bmatrix} x + \begin{bmatrix} 2 \\ 3 \\ 1 \\ 2 \\ 3 \end{bmatrix} u$$

- $\ell(x,u) = x^T x + u^2$; $F(x) = x^T x$; $\mathbb{X}_F = \{0\}$; $\mathbb{U} = [-40, 40]$
- $x_0 = [1,1,1,1,1]^T$

Without terminal constraints:

- Feasibility is guaranteed for all $N \in \mathbb{N}$
- V_N is only strictly decreasing for $N \geq 10$

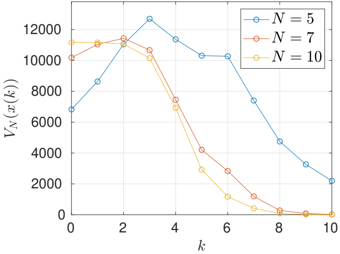
With terminal constraints:

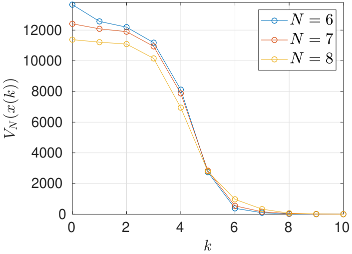
- Feasibility only guaranteed for $N \geq 6$
- V_N is strictly decreasing for all $N \geq 6$ (as expected)

Note that:

- Here, we only look at one initial condition!
- The observations are not necessarily satisfied for all x_0!

Open loop costs $V_N(x(k))$ without terminal constraints and with terminal constraints





Viability & Recursive Feasibility

Note that:

- If $\mathbb{X} \neq \mathbb{R}^n$ then the OCP may be infeasible.
- To define implementable feedback laws it is necessary that the OCP is feasible for all $k \in \mathbb{N}$.

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Definition (Viability)

Consider $x^+ = f(x,u)$ together with $\mathbb{X} \subset \mathbb{R}^n$ and $\mathbb{U}(x) \subset \mathbb{R}^m$ for all $x \in \mathbb{X}$. The set $\mathbb{X}$ is called viable if

$$\forall\, x \in \mathbb{X} \quad \exists\, u \in \mathbb{U}(x) \ \text{ such that } \ f(x,u) \in \mathbb{X}.$$

A viable set $\mathbb{X}$ is also called a *control invariant set*.

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- Consider $x = \operatorname{sign}(a)$.
- For $u = 0$, x^+ satisfies $x^+ = a \operatorname{sign}(a) = |a| > 2$.
- The best we can is to select $u = -1$. Thus $x^+ = a \operatorname{sign}(a) - 1 = |a| - 1 > 1$

$\rightsquigarrow$ For $|a| > 2$, the set $\mathbb{X}$ is not viable.

Viability & Recursive Feasibility (2)

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Moreover, the selection of $u = -1$ implies that

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- For $x \leq 0$ the same arguments (with $u = 1$) lead to $c_1 = c_2$.

⇝ The maximal viable set contained in $\mathbb{X}$ is given by $\mathbb{X}_\mathrm{v} = [-\frac{1}{2}, \frac{1}{2}]$.

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- For higher dimensional linear systems subject to constraints $\mathbb{X}$ and $\mathbb{U}$ the verification of viability quickly becomes tedious

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and $\mathbb{X} = [-1, 1]^2$ and $\mathbb{U} = [-\frac{1}{4}, \frac{1}{4}]$.

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Idea: Starting with $\mathbb{X}_0 = \mathbb{X}$ iteratively define

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$$\widetilde{\mathbb{X}}_i = \{[x^T, u]^T \in \mathbb{R}^3 : \Delta_i (x^T, u)^T \leq \delta_i\} \quad (1)$$

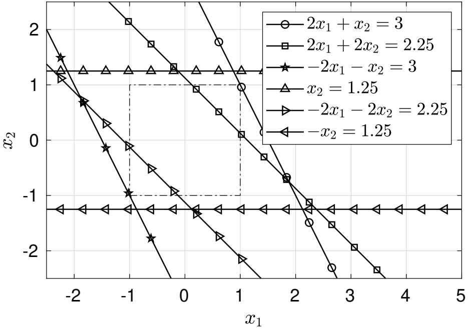
with

$$\Delta_i = \begin{bmatrix} \Gamma_i & 0 \\ \Gamma_i A & \Gamma_i B \\ 0 & \Gamma_u \end{bmatrix} \quad \text{and} \quad \delta_i = \begin{bmatrix} \gamma_i \\ \gamma_i \\ \gamma_u \end{bmatrix}.$$

Then, $\mathbb{X}_{i+1} = P_x(\widetilde{\mathbb{X}}_i)$ is obtained by projecting $\widetilde{\mathbb{X}}_i$ on the (x_1, x_2)-subspace.

Viability & Recursive Feasibility (4)

The projection $\mathbb{X}_1 = P_x(\widetilde{\mathbb{X}}_0)$ leads to the conditions represented in the figure:



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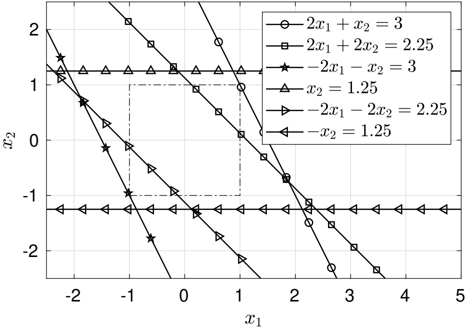
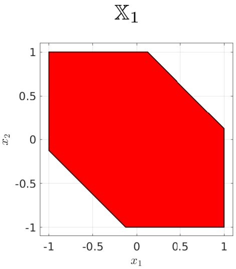
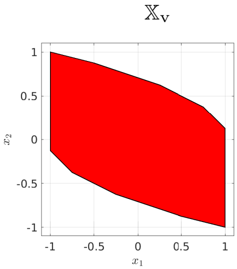


(⇝ Have a look in the lecture notes for details.)





The viable set $\mathbb{X}_{\mathrm{v}}$ is defined through:

$$\begin{bmatrix} -0.24 & -0.97 \\ -0.32 & -0.95 \\ 0.71 & 0.71 \\ 0.32 & 0.95 \\ -0.45 & -0.89 \\ 0.24 & 0.97 \\ 0.45 & 0.89 \\ -0.71 & -0.71 \\ 1.00 & 0.00 \\ -1.00 & 0.00 \end{bmatrix} x \leq \begin{bmatrix} 0.73 \\ 0.67 \\ 0.80 \\ 0.67 \\ 0.67 \\ 0.73 \\ 0.67 \\ 0.80 \\ 1.00 \\ 1.00 \end{bmatrix}$$

Viability & Recursive Feasibility (4)

Definition (Recursive feasibility)

Consider the MPC Algorithm with input constraints $\mathbb{U}$ and a set of initial states $\mathbb{X}_{\mathrm{rf}}^N \subset \mathbb{X}$. The set $\mathbb{X}_{\mathrm{rf}}^N$ is called recursively feasible with respect to the MPC Algorithm and the prediction horizon $N \in \mathbb{N}$ if feasibility of the OCP for $x(0) = x_0 \in \mathbb{X}_{\mathrm{rf}}^N$ implies feasibility of the OCP for all $k \in \mathbb{N}$.

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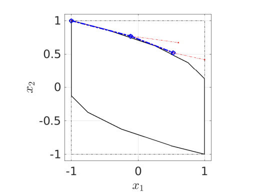
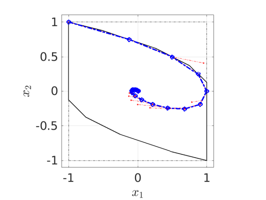
Example (continued)

- For $N = 3$ we arrive at an infeasible OCP after 2 iterations (left)
- For $N = 4$ (and in fact also for $N \geq 4$), the MPC algorithm does not run into an infeasible optimization problem (right).





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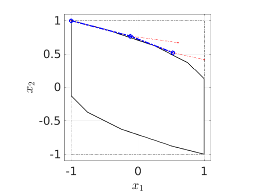

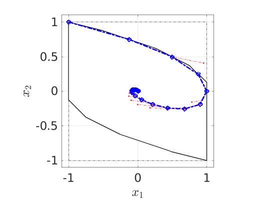

Note that:

- If we replace $\mathbb{X}$ by $\mathbb{X}_{\text{v}}$ in the MPC algorithm, then infeasibility is not a problem. (However, this means we need to know $\mathbb{X}_{\text{v}}$.)

Viability & Recursive Feasibility (5)

Note that:

- Recursive feasibility shifts the problem of running into an infeasible optimization problem from viability to recursive feasibility. However, similar to viability, recursive feasibility of a set $\mathbb{X}_{\mathrm{rf}}^N$ is in general nontrivial to establish.

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Consider the MPC Algorithm and assume that $\mathbb{U}(x) = \mathbb{U} \subset \mathbb{R}^m$ for all $x \in \mathbb{X}$ and $\mathbb{X}_F = \mathbb{R}^n$. If $\mathbb{X}$ is viable, then $\mathbb{X}_{\mathrm{rf}}^N = \mathbb{X}$ is recursively feasible for all $N \in \mathbb{N}$.

Proof.

- Since $\mathbb{X}$ is viable, for all $x(k) \in \mathbb{X}$ there exist $u(k) \in \mathbb{U}$ such that $x(k+1) \in \mathbb{X}$, $k = 0, \ldots, N-1$.
- If $x(0) = x_0 \in \mathbb{X}$ is satisfied, the OCP is feasible.
- At the next time step, the OCP is initialized through $f(x_0, u^\star(0; x_0)) \in \mathbb{X}$ and the same argument can be applied iteratively.

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Lemma

Consider the MPC Algorithm and assume that $\mathbb{U}(x) = \mathbb{U} \subset \mathbb{R}^m$ *for all* $x \in \mathbb{X}$*. Additionally assume that* $\mathbb{X}_F \subset \mathbb{X}$ *defines a viable set. If the OCP is feasible for all* $x_0 \in \mathbb{X}_{\mathrm{rf}}^N$*, then* $\mathbb{X}_{\mathrm{rf}}^N = \mathbb{X}$ *is recursively feasible.*

Proof.

- Let the OCP be feasible for all $x_0 \in \mathbb{X}_{\mathrm{rf}}^N$.
- Then there exist $u(k) \in \mathbb{U}$ such that $x(k+1) \in \mathbb{X}$ for all $k = 0, \ldots, N-1$ and $x(N) \in \mathbb{X}_F$.
- Moreover, since $\mathbb{X}_F$ is viable, there exists $u(N) \in \mathbb{U}$ such that $x(N+1) \in \mathbb{X}_F$. In particular $u(1), \ldots, u(N)$ is feasible for the OCP at time $k = 1$ initialized through $x_0 = x(1)$.
- This argument can be applied iteratively showing recursive feasibility.

□

Hard and Soft Constraints

Note that:

- Infeasibility of the OCP can only occur in the presence of state constraints $\mathbb{X} \neq \mathbb{R}^n$.
- In some applications it is justifiable to circumvent infeasible optimization problems by rewriting *hard constraints* as *soft constraints*.

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Recall

- The combined state and input constraints: $\mathbb{D} \subset \mathbb{R}^{n+m}$

Define

- Distance to $\mathbb{D}$:

$$d_{\mathbb{D}}(x,u) = \min_{(v,w)\in\mathbb{D}} \sqrt{|x-v|^2 + |u-w|^2}$$

- Distance to the terminal set $\mathbb{X}_F$: $d_F : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$,

$$d_F(x) = \min_{v\in\mathbb{X}_F} |x-v|$$

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$$\mathcal{U}^N = \left\{ u_N(\cdot) \;\middle|\; \begin{array}{rcl} x(0) & = & x_0, \\ x(k+1) & = & f(x(k),u(k)) \\ & \forall & k \in \mathbb{N}_{[0,N-1]} \end{array} \right\}$$

- OCP:

$$V_N(x_0) = \min_{u_N(\cdot)\in\mathcal{U}^N} J_N(x_0,u_N(\cdot)) + F(x(N)) + \sum_{i=0}^{N-1} \ell_{\mathsf{s}}(x(i),u(i)) + F_{\mathsf{s}}(x(N))$$

$$\text{subject to } x^+ = f(x,u),\ x(0) = x_0,$$

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Note that:

- ⇝ A solution doesn't necessarily satisfy the constraints
- ⇝ The OCP is feasible by construction
- ⇝ $(x,u) \in \mathbb{D}$, $x \in \mathbb{X}_F$ are *hard constraints*, while $\ell_{\mathsf{s}}(x,u)$, $F_{\mathsf{s}}(x)$ in the cost function define *soft constr.*
- ⇝ If $(x,u) \in \mathbb{D}$ & $x \in \mathbb{X}_F$ then $\ell_{\mathsf{s}}(x,u) = 0$ & $F_{\mathsf{s}}(x) = 0$
- ⇝ If $(x,u) \notin \mathbb{D}$ & $x \notin \mathbb{X}_F$ then $\ell_{\mathsf{s}}(x,u) > 0$ & $F_{\mathsf{s}}(x) > 0$ impose additional costs.

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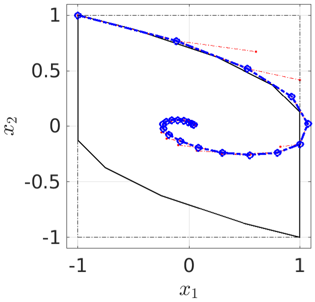
Rewrite hard constraints into soft constraints:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} x - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} s \leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Penalize $10000 s(i)^T s(i)$ in the cost function



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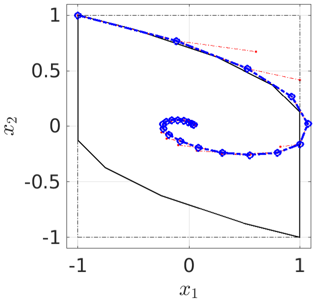

Note that, alternatively the constraints

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with cost function $10000 s^2$ could have been used, for example.

Section 2

Model Predictive Control Schemes

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Model Predictive Control Schemes: (not a comprehensive list)

- MPC for Time-Varying Systems & Reference Tracking
- Linear MPC
- Nonlinar MPC
- MPC Without Terminal Costs & Constraints (a.k.a. unconstrained MPC)
- Explicit MPC
- Economic MPC
- Robust MPC
- Tube Based MPC
- Stochastic MPC
- Chance constraint MPC
- Distributed MPC
- Multi-step MPC

Model Predictive Control Schemes: Time-Varying Systems & Reference Tracking

Consider: $f : \mathbb{N}_0 \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$

$$x(k+1) = f(k, x, u), \qquad x(k_0) = x_0 \in \mathbb{R}^n, \; k_0 \in \mathbb{N}_0$$

Model Predictive Control Schemes: Time-Varying Systems & Reference Tracking

Consider: $f : \mathbb{N}_0 \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$

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- Time-varying sets/constraints

$$\mathbb{X}(k) \subset \mathbb{R}^n \quad \text{and} \quad \mathbb{U}(k,x) \subset \mathbb{R}^m \quad \forall k \in \mathbb{N}_0$$
$$\mathbb{D}(k) = \mathbb{X}(k) \times \mathbb{U}(k,x) \subset \mathbb{R}^n \times \mathbb{R}^m \qquad \forall\, k \in \mathbb{N}_0$$

- Set of feasible input trajectories

$$\mathcal{U}_{\mathbb{D}}^N(k) = \left\{ u_N(\cdot;k) : \mathbb{N}_{[k,k+N-1]} \to \mathbb{R}^m \;\middle|\; \begin{array}{r} x(k) = x_0, \\ x(i+1) = f(i, x(i), u(i)) \\ (x(i), u(i)) \in \mathbb{D}(i), \\ \forall\, i \in \mathbb{N}_{[k,k+N-1]} \end{array} \right\}$$

- Cost function & running cost:

$J_N : \mathbb{N}_0 \times \mathbb{R}^n \times \mathcal{U}_{\mathbb{D}}^N(k) \to \mathbb{R} \cup \{\infty\}, \quad \ell : \mathbb{N}_0 \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$

$$J_N(k, x_0, u_N(\cdot)) = \sum_{i=k}^{k+N-1} \ell(i, x(i), u(i))$$

- Terminal cost & terminal constraints

$$F : \mathbb{N}_0 \times \mathbb{R}^n \to \mathbb{R} \qquad \mathbb{X}_F(k) \subset \mathbb{R}^n, \qquad \forall k \in \mathbb{N}_0$$

- Optimal control problem:

$$V_N(k, x_0) = \min_{u_N(\cdot;k) \in \mathcal{U}_{\mathbb{D}}^N(k)} J_N(k, x_0, u_N(\cdot)) + F(k, x(N))$$
$$\text{s.t. dynamics } \& \ x(N) = \mathbb{X}_F(k)$$

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Input: $k_0 \in \mathbb{N}_0$; $x(k_0)$; $N \in \mathbb{N} \cup \{\infty\}$; $\ell : \mathbb{N}_0 \times \mathbb{R}^{n+m} \to \mathbb{R}$; $\mathbb{D}(k) \subset \mathbb{R}^{n+m}$; $F : \mathbb{N}_0 \times \mathbb{R}^n \to \mathbb{R}$; $\mathbb{X}_F(k) \subset \mathbb{R}^n$.

For $k = k_0, k_0+1, k_0+2, \ldots$

1. Measure the current state and define $x_0 = x(k)$.
2. Solve OCP to obtain open-loop control law $u_N^\star(\cdot; k, x_0)$.
3. Define the feedback law $\mu_N(k, x(k)) = u_N^\star(0; k, x_0)$
4. $x(k+1) = f(k, x(k), \mu_N(k, x(k)))$ increment k to $k+1$ and go to 1.

Model Predictive Control Schemes: Time-Varying Systems & Reference Tracking

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Typical Running costs for reference tracking

$$\ell(k, x, u) = (x - x_{\text{ref}}(k))^T Q (x - x_{\text{ref}}(k)) + (u - u_{\text{ref}}(k))^T R (u - u_{\text{ref}}(k))$$
$$Q \in \mathcal{S}^n_{\geq 0},\ R \in \mathcal{S}^m_{\geq 0}$$

Model Predictive Control Schemes: Linear & Nonlinear MPC

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Example (Quadratic Program)

For linear dynamics $x^+ = Ax + Bu$, $Q, P \in \mathcal{S}^n_{\geq 0}$, $R \in \mathcal{S}^m_{\geq 0}$ and polyhedral constraints defined through $\Gamma_x \in \mathbb{R}^{r\times n}$, $\Gamma_u \in \mathbb{R}^{r\times m}$, $\gamma \in \mathbb{R}^r$, $\Gamma_N \in \mathbb{R}^{q\times n}$, $\gamma_N \in \mathbb{R}^q$.
OCP can be written as a QP of the form

$$\min_{\substack{u(i)\in\mathbb{R}^m \\ i\in\mathbb{N}_{[0,N-1]}}} \sum_{i=0}^{N-1} x(i)^T Q x(i) + u(i)^T R u(i) + x(N)^T P x(N)$$

$$\begin{array}{llcll}
\text{subject to} & 0 & = & x(0) - x_0 & \\
& 0 & = & x(i+1) - Ax(i) - Bu(i) & \forall\, i \\
& \gamma & \geq & \Gamma_x x(i) + \Gamma_u u(i) & \forall\, i \\
& \gamma_N & \geq & \Gamma_N x(N) &
\end{array}$$

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Example (Convex programs)

Terminal constraints based on a quadratic Lyapunov function, i.e., ($P \in \mathcal{S}^n_{>0}$, $c \in \mathbb{R}_{>0}$)

$$\mathbb{X}_F = \{x \in \mathbb{R}^n : \ x^T P x \leq c\}$$

Convex running cost:

$$\ell(x,u) = (x^T x)^2 + (u^T u)^2$$

Convex optimization problem:

$$\min_{\substack{u(i) \in \mathbb{R}^m \\ i \in \mathbb{N}_{[0,N-1]}}} \sum_{i=0}^{N-1} (x(i)^T x(i))^2 + (u(i)^T u(i))^2$$

$$\begin{array}{lccll} \text{subject to} & 0 & = & x(0) - x_0 & \\ & 0 & = & x(i+1) - Ax(i) - Bu(i) & \forall\, i \\ & \gamma & \geq & \Gamma_x x(i) + \Gamma_u u(i) & \forall\, i \\ & c & \geq & x(N)^T P x(N). & \end{array}$$

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Example (Nonlinear optimization)

Inverted pendulum on a cart

$$\min_{\substack{u(i)\in\mathbb{R}^m\\ i\in\mathbb{N}_{[0,N-1]}}} \sum_{i=0}^{N-1} x_1^2 + (1-\cos(x_2))^2 + x_3^2 + x_4^2 + u^2$$

$$\begin{array}{llll}
\text{subject to} & 0 = & x(0) - x_0 & \\
 & 0 = & x(i+1) - x(i) - \Delta f(x(i),u(i)) & \forall\, i \\
 & c_u \geq & u(i) & \forall\, i \\
 & c_u \geq & -u(i) & \forall\, i \\
 & c_x \geq & x_1(i) & \forall\, i \\
 & c_x \geq & -x_1(i) & \forall\, i
\end{array}$$

Here $\dot{x} = f(x,u) =$

$$\begin{bmatrix} x_3 \\ x_4 \\ \frac{-\bar{J}\bar{c}x_3 - \bar{J}\sin(x_2)x_4^2 - \bar{\gamma}\cos(x_2)x_4 + g\cos(x_2)\sin(x_2) + \bar{J}u}{\bar{M}\bar{J} - \cos^2(x_2)} \\ \frac{-\bar{M}\bar{\gamma}x_4 + \bar{M}g\sin(x_2) - \bar{c}\cos(x_2)x_3 - \cos(x_2)\sin(x_2)x_4^2 + \cos(x_2)u}{\bar{M}\bar{J} - \cos^2(x_2)} \end{bmatrix}$$

Model Predictive Control Schemes: MPC Without Terminal Costs & Constraints

Infinite horizon optimal control problem

$$V_\infty(x_0) = \min_{u_\infty(\cdot)\in\mathcal{U}_\mathbb{D}^\infty} J_\infty(x_0, u(\cdot))$$

subject to dyn. & init. cond.

Finite horizon optimal control problem

$$V_N(x_0) = \min_{u_N(\cdot)\in\mathcal{U}_\mathbb{D}^N} J_N(x_0, u_N(\cdot)) + F(x(N))$$

subject to dyn. & init. cond. and $x(N) \in \mathbb{X}_F$

- Often, a finite horizon OCP is used to approximate a corresponding infinite horizon OCP

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⇝ Compromise in numerical complexity versus optimality

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$\rightsquigarrow$ Compromise in numerical complexity versus optimality

- What is a “good” approximation of the infinite horizon OCP?
- How to select N, $\mathbb{X}_F$ and F?
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- Are terminal costs/constraints necessary?
  - ‘For sufficiently large prediction horizons $N \in \mathbb{N}$, the MPC closed loop without terminal costs/constraints approximates the corresponding infinite horizon solution arbitrarily well.’
  - Found in publications on ‘MPC Without Terminal Costs & Constraints’ or ‘Unconstrained MPC’

Model Predictive Control Schemes: Explicit MPC

For $k = 0, 1, 2, \ldots$

1. Measure the current state of the system $x^+ = f(x, u)$ and define $x_0 = x(k)$.
2. Solve OCP $\rightsquigarrow$ open-loop input $u_N^\star(\cdot; x_0)$
3. Define $\mu_N(x(k)) = u_N^\star(0; x_0)$
4. Compute $x(k+1) = f(x(k), \mu_N(x(k)))$, increment k to $k+1$ and go to 1.

Note that:

- At every time step an optimization problem needs to be solved
- The optimal value function and the optimal feedback law is only known implicitly

However:

- In some cases it is possible to compute an *explicit* solution of the OCP as a function of x_0.

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Example

Consider $x^+ = x + 0.5u$ with $u \in [-1, 1]$.

- The origin is stable but not asymptotically stable.
- Every state $x_0 \in \mathbb{R}$ can be driven to the origin in finite time.
- OCP with $\ell(x,u) = x^2 + u^2$ and $N = 2$:

$$\min_{u(0),u(1)} x(0)^2 + x(1)^2 + u(0)^2 + u(1)^2$$

$$\begin{aligned} \text{subject to} \quad 0 &= x(0) - x_0 \\ 0 &= x(1) - x(0) - 0.5u(0) \\ 0 &= x(2) - x(1) - 0.5u(1) \\ u(0), u(1) &\in [-1, 1]. \end{aligned}$$

- Equivalently

$$\min_{u(0),u(1)} 2x_0^2 + x_0 u(0) + 1.25u(0)^2 + u(1)^2$$

$$\text{subject to} \quad u(0), u(1) \in [-1, 1].$$

Model Predictive Control Schemes: Explicit MPC (2)

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- The gradient of the objective function

$$J_2(u(\cdot); x_0) = 2x_0^2 + x_0 u(0) + 1.25u(0)^2 + u(1)^2$$

is given by

$$\nabla_u J_2(u(\cdot); x_0) = \begin{bmatrix} 2.5u(0) + x_0 \\ 2u(1) \end{bmatrix} \stackrel{!}{=} 0$$

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Model Predictive Control Schemes: Explicit MPC (2)

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- The optimal value function: (cont. differentiable)

$$V_2(x_0) = \begin{cases} 2x_0^2 + x_0 + 1.25 & \text{if} \quad x_0 \leq -2.5 \\ 1.8x_0^2 & \text{if} \quad x_0 \in [-2.5, 2.5] \\ 2x_0^2 - x_0 + 1.25 & \text{if} \quad x_0 \geq 2.5 \end{cases}$$

Model Predictive Control Schemes: Explicit MPC (3)

Example (More general setting)

Consider the linear system $x^+ = Ax + Bu$ defined through

$$A = \begin{bmatrix} 1 & 1 \\ -\frac{1}{4} & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

- Constraints: $x \in [-5, 5]^2$, $u \in [-1, 1]$; Horizon: $N = 5$.
- Running cost, terminal cost: $\ell(x, u) = x^T Q x + u^T R u$, $F(x) = x^T P x$,

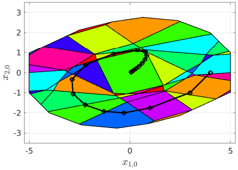
$$Q = P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad R = 1,$$

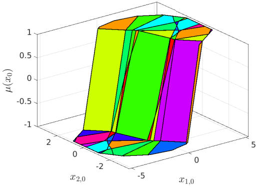
⇝ OCP is a quadratic program

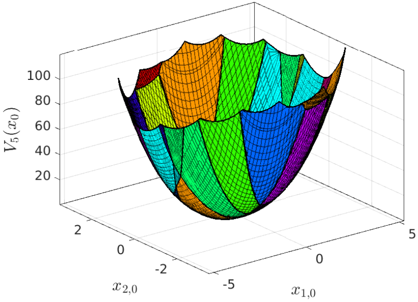
⇝ Feasible region is convex (partition in 53 polyhedral sets)

⇝ $\mu_5(x_0)$ is continuous and piecewise affine

⇝ $V_5(x_0)$ is continuously differentiable







Model Predictive Control Schemes: Economic MPC

Note that:

- So far we have tacitly assumed that the running cost ℓ is a positive semidefinite function penalizing the distance to a reference trajectory $(x_{\text{ref}}(t), u_{\text{ref}}(t))$.

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OCP:

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- The optimal average cost satisfies

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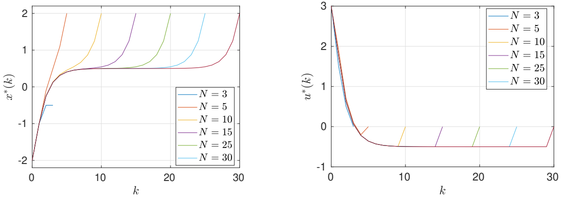
Open-loop solutions for different N



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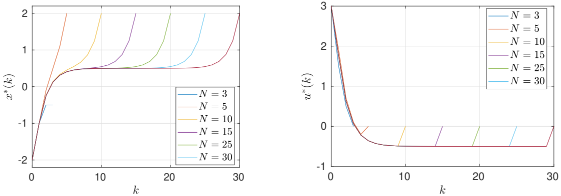
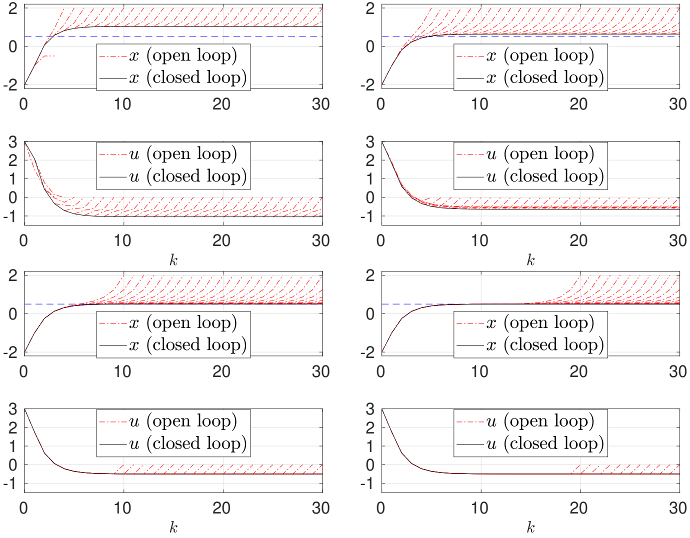


Turnpike property

- an approaching arc, converging to ≈ 0.5;
- a stable segment, staying at ≈ 0.5;
- a leaving arc, diverging from ≈ 0.5.

Model Predictive Control Schemes: Economic MPC

Open-loop and closed-loop solutions with respect to $N \in \{3, 5, 10, 20\}$



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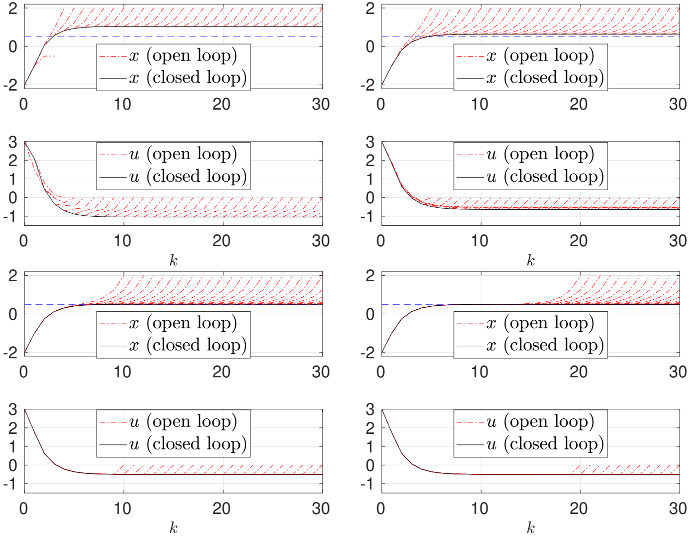


Optimal steady-state (x^e, u^e):
Can be calculated through

$$\min \; u^2 - x$$
$$\text{s.t. } x = x^+ = 2x + u$$

(but it is not necessary)

Alternatively, the running costs

$$\tilde{\ell}(x, u) = c_1|x - x^e|^2 + c_2|u - u^e|^2$$

asymptotically lead to the same closed-loop solution

However, the transient behavior is different

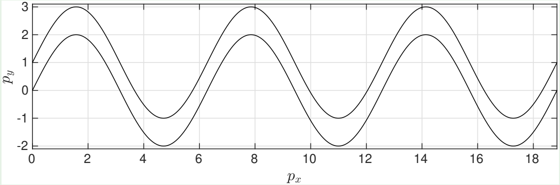
Model Predictive Control Schemes: Economic MPC

Example (Mobile Robot)

Continuous and discrete time dynamics ($\Delta > 0$):

$$\begin{bmatrix} \dot{p}_x \\ \dot{p}_y \\ \dot{\phi} \\ \dot{v} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} v\cos(\phi) \\ v\sin(\phi) \\ w \\ a \\ q \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} p_x^+ \\ p_y^+ \\ \phi^+ \\ v^+ \\ w^+ \end{bmatrix} = \begin{bmatrix} p_x + \Delta v\cos(\phi) \\ p_y + \Delta v\sin(\phi) \\ \phi + \Delta w \\ v + \Delta a \\ w + \Delta q \end{bmatrix}$$

Goal: Finish the "race track" as quick as possible



The track is defined through the set

$$\mathcal{P} = \{[p_x, p_y]^T \in \mathbb{R}^2 | p_y \geq 2\sin(p_x), \;\; p_y \leq 2\sin(p_x) + 1\}$$

Additional constraints: $p_x(t_1) \leq p_x(t_2)$ for all $t_1 \leq t_2$.
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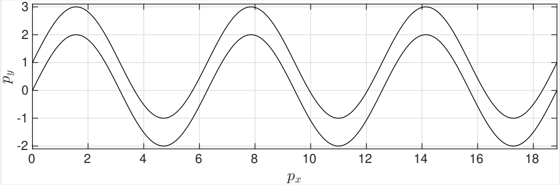

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Maximize the distance traveled in p_x-direction:

$$\max \int_0^{T_{end}} v(t)\cos(\phi(t))dt = -\min \int_0^{T_{end}} -v(t)\cos(\phi(t))dt$$

$$\max \Delta \sum_{i=0}^{K_{end}} v(i)\cos(\phi(i)) = -\min \Delta \sum_{i=0}^{K_{end}} -v(i)\cos(\phi(i))$$

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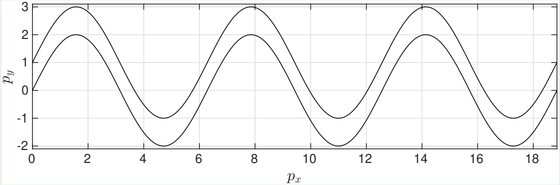
Goal: Finish the "race track" as quick as possible



The track is defined through the set

$$\mathcal{P} = \{[p_x, p_y]^T \in \mathbb{R}^2 | p_y \geq 2\sin(p_x), \;\; p_y \leq 2\sin(p_x) + 1\}$$

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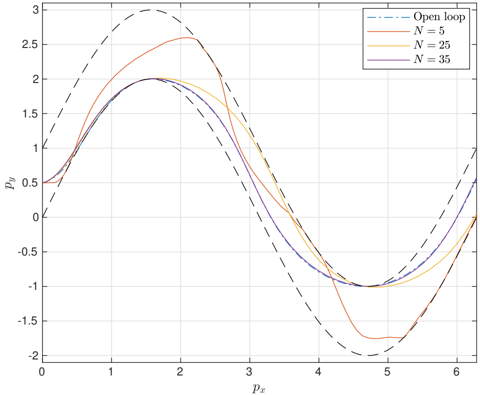

Model Predictive Control Schemes: Economic MPC

Example (Mobile Robot)

Continuous and discrete time dynamics ($\Delta > 0$):

$$\begin{bmatrix} \dot{p}_x \\ \dot{p}_y \\ \dot{\phi} \\ \dot{v} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} v\cos(\phi) \\ v\sin(\phi) \\ w \\ a \\ q \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} p_x^+ \\ p_y^+ \\ \phi^+ \\ v^+ \\ w^+ \end{bmatrix} = \begin{bmatrix} p_x + \Delta v\cos(\phi) \\ p_y + \Delta v\sin(\phi) \\ \phi + \Delta w \\ v + \Delta a \\ w + \Delta q \end{bmatrix}$$

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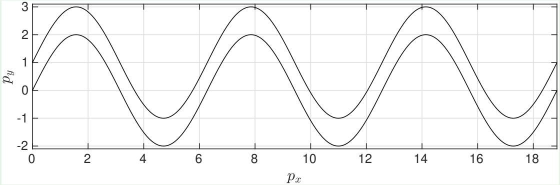

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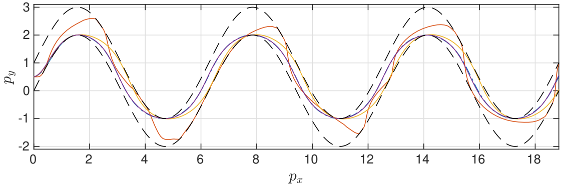

Model Predictive Control Schemes: Economic MPC

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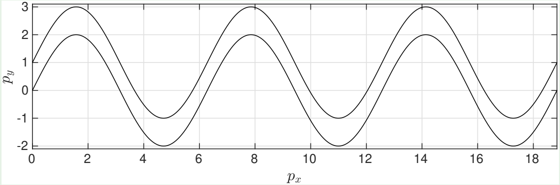

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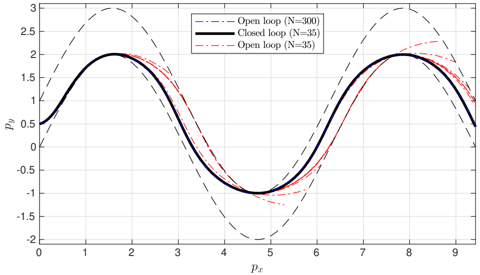

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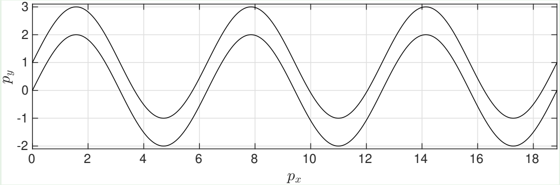

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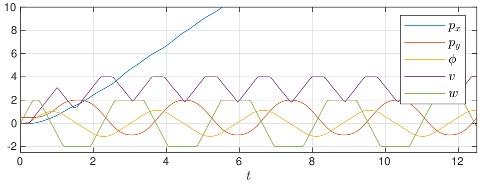
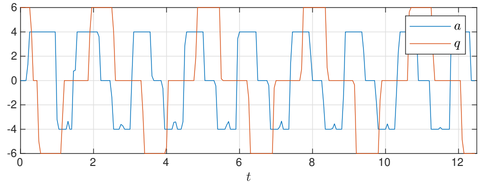


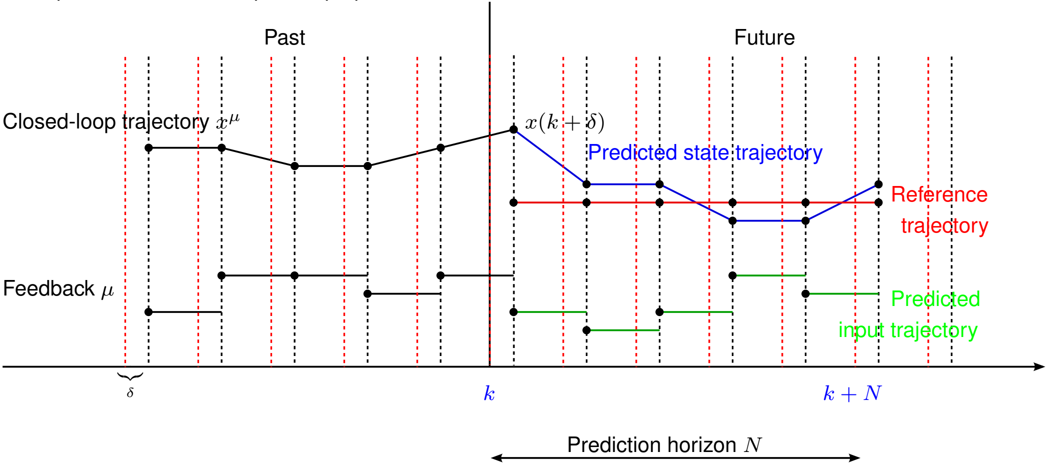


Section 3

Implementational Aspects of MPC

Implementational Aspects of MPC

So far, we have implicitly assumed that the OCP can be solved instantaneously
$\rightsquigarrow$ Introduce $\delta > 0$ as an upper bound for the time to solve the OCP, or
$\rightsquigarrow$ use multiple elements of the open-loop optimal solution to define the feedback law



Warm-Start

Optimal open loop input and state trajectories at time $k \in \mathbb{N}$ with respect to the initial state x_0:

$$u^\star(\cdot; k, x_0) = \begin{bmatrix} u_N^\star(0; k, x_0) \\ \vdots \\ u_N^\star(N-1; k, x_0) \end{bmatrix}, \qquad x^\star(\cdot; k, x_0) = \begin{bmatrix} x_N^\star(0; k, x_0) \\ \vdots \\ x_N^\star(N-1; k, x_0) \end{bmatrix}$$

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If the prediction horizon N is large, it is not unreasonable to assume that

$$u^\star(i+1; k, x_0) \approx u^\star(i; k+1, x^\star(1; k, x_0)), \qquad x^\star(i+1; k, x_0) \approx x^\star(i; k+1, x^\star(1; k, x_0))$$

is satisfied for $i = 0, \dots, N-2$.

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The initialization

$$u^0(\cdot; k+1, x^\star(1; k, x_0)) = \begin{bmatrix} u_N^\star(1; k, x_0) \\ \vdots \\ u_N^\star(N-1; k, x_0) \\ 0 \end{bmatrix} \qquad x^0(\cdot; k+1, x^\star(1; k, x_0)) = \begin{bmatrix} x_N^\star(1; k, x_0) \\ \vdots \\ x_N^\star(N-1; k, x_0) \\ f(x_N^\star(N-1; k, x_0), 0) \end{bmatrix}$$

CAN reduce the numerical complexity at the next time step $k+1$ significantly.

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CAN reduce the numerical complexity at the next time step $k+1$ significantly.

Remark

If the OCP is non-convex and has multiple local minima, warm-start may be counterproductive in finding a global minimum.

Formulation of the Optimization Problem

Standard formulation of an optimization problem:

$$y^\star = \arg \min_{y \in \mathbb{R}^{\alpha_1}} \; F(y)$$

$$\begin{aligned} \text{s.t.} \quad & G_i(y) \leq 0, \qquad & i = 1, \ldots, \alpha_2 \\ & H_j(y) = 0, \qquad & j = 1, \ldots, \alpha_3 \end{aligned}$$

Different possibilities to define the unknown y:
Option 1: (Full discretization)

$$y = \begin{bmatrix} x(0)^T & u(0)^T & \cdots & x(N-1)^T & u(N-1)^T & x(N)^T \end{bmatrix}^T.$$

- More unknowns, larger number of constraints, but sparsity patterns

Option 2: (Recursive elimination)

$$y = \begin{bmatrix} u(0)^T & \cdots & u(N-1)^T \end{bmatrix}^T,$$

- Smaller number of unknowns, smaller number of constraints, but dense representations

Also see:

- Single shooting & multiple shooting

(Depending on the optimization algorithm, different representations have advantages/disadvantages)

Introduction to Nonlinear Control

Stability, control design, and estimation

Philipp Braun & Christopher M. Kellett
School of Engineering,
Australian National University, Canberra, Australia

Part II:

Chapter 15: Model Predictive Control

Australian National University