Introduction to Nonlinear Control

Stability, control design, and estimation

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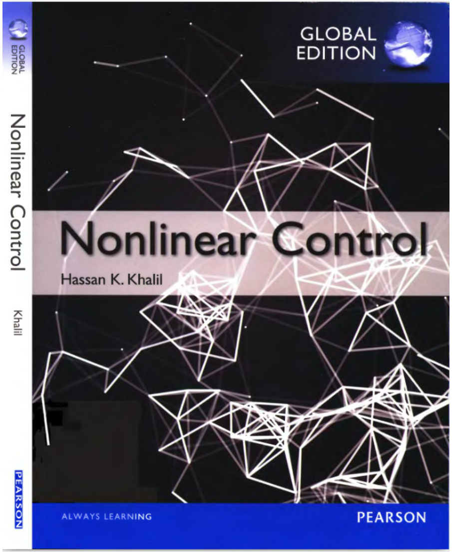
Part IV:

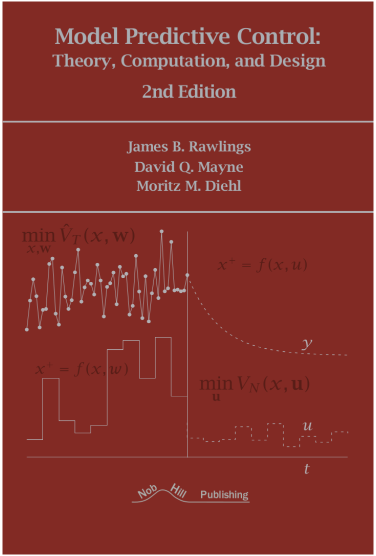
Chapter 17: The Extended Kalman Filter

17.1 Extended Kalman Filter (Continuous Time)
17.2 Extended Kalman Filter (Discrete Time)
17.3 Moving Horizon Estimation

Australian National University

The Extended Kalman Filter





Nonlinear Systems - Fundamentals

Section 1

Extended Kalman Filter (Continuous time)

Extended Kalman Filter (Continuous time)

Consider (nonlinear) system with (nonlinear) output:

$$\dot{x}(t) = f(x(t), u(t)), \qquad x(0) \in \mathbb{R}^n$$
$$y(t) = h(x(t)).$$

Assumptions & Notations:

- state $x \in \mathbb{R}^n$; input $u \in \mathbb{R}^m$; measured output $y \in \mathbb{R}^p$
- $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^p$ arbitrarily often continuously differentiable by assumption.
- For cont. $u(\cdot)$, $f(x, u(t))$ satisfies

$$A(t) = \left[\frac{\partial f(x(t), u(t))}{\partial x}\right]_{x=0}; \qquad \lim_{|x|\to 0} \sup_{t\geq 0} \frac{|f(t,x) - A(t)x|}{|x|} = 0$$

$\rightsquigarrow$ 'The linearization of f makes sense'

Observer dynamics:

$$\dot{\hat{x}}(t) = f(\hat{x}(t), u(t)) + L(t)(y(t) - h(\hat{x}(t)))$$

- $\hat{x} \in \mathbb{R}^n$: estimated state
- $L : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n\times p}$ represents a time-dependent output injection term

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Observer design:

- Error $e = x - \hat{x}$ and error dynamics

$$\dot{e} = f(x, u) - f(\hat{x}, u) - L(t)(h(x) - h(\hat{x}))$$

- Define (time-varying linearization in $(\hat{x}, u)$)

$$A(t) = \frac{\partial f}{\partial x}(\hat{x}(t), u(t)) \qquad \text{and} \qquad C(t) = \frac{\partial h}{\partial x}(\hat{x}(t))$$

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- Adding and subtracting $(A(t) - L(t)C(t))e$ we obtain

$$\dot{e} = (A(t) - L(t)C(t))e + \Delta(e, x, u) \qquad (1)$$

where

$$\Delta(e, x, u) = f(x, u) - f(\hat{x}, u) - A(t)e - L(t)(h(x) - h(\hat{x}) - C(t)e)$$

- Note that: $f(e + \hat{x}, u) - f(\hat{x}, u) = 0$ for $e = 0$ and

$$\frac{\partial}{\partial e}(f(x, u) - f(\hat{x}, u))\Big|_{e=0} = \frac{\partial}{\partial e}(f(e + \hat{x}, u) - f(\hat{x}, u))\Big|_{e=0} = \frac{\partial}{\partial e}f(e + \hat{x}, u)\Big|_{e=0} = A(t)$$

$$\frac{\partial}{\partial e}(h(x) - h(\hat{x}))\Big|_{e=0} = \frac{\partial}{\partial e}h(e + \hat{x})\Big|_{e=0} = C(t)$$

⇝ (1) represents Taylor approximation at $e = 0$ w.r.t. $\hat{x}$

Extended Kalman Filter (Continuous time) (2)

Derivation of a time-dependent injection gain $L(t)$:

- Consider $P(t) > 0$ and $\alpha_1, \alpha_2 \in \mathbb{R}_{>0}$ such that

$$\alpha_1 I \le P(t) \le \alpha_2 I, \quad \frac{1}{\alpha_2} I \le P^{-1}(t) \le \frac{1}{\alpha_1} I \quad \forall\, t$$

- Candidate Lyapunov function $V : \mathbb{R} \times \mathbb{R}^p \to \mathbb{R}_{\ge 0}$

$$V(e(t)) = e(t)^T P^{-1}(t) e(t)$$

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Lemma

Consider $P : \mathbb{R} \to \mathcal{S}^n_{>0}$ *cont. diff. Then*

$$\dot{P}^{-1}(t) = -P^{-1}(t)\dot{P}(t)P^{-1}(t).$$

Proof.

With $I = P(t)P^{-1}(t) = P^{-1}(t)P(t)$ it holds that

$$\begin{aligned}\dot{P}^{-1} &= \tfrac{d}{dt}\left(P^{-1}PP^{-1}\right)\\ &= \dot{P}^{-1}PP^{-1} + P^{-1}\dot{P}P^{-1} + P^{-1}P\dot{P}^{-1}\\ &= 2\dot{P}^{-1} + P^{-1}\dot{P}P^{-1}\end{aligned}$$

□

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\end{aligned}$$

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Derivative of the candidate Lyapunov function:

$$\begin{aligned}
\dot{V}(e) &= \dot{e}^T P^{-1} e + e^T \dot{P}^{-1} e + e^T P^{-1} \dot{e} \\
&= ((A - LC)e + \Delta)^T P^{-1} e + e^T P^{-1}((A - LC)e + \Delta) - e^T P^{-1} \dot{P} P^{-1} e \\
&= e^T P^{-1} \left(P(A - LC)^T + (A - LC)P - \dot{P}\right) P^{-1} e + 2e^T P^{-1} \Delta
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Select $L(t) = P(t)C(t)^T Q$ for $Q > 0$. Then:

$$\dot{V}(e) = e^T P^{-1}\left(PA^T + AP - 2PC^T QCP - \dot{P}\right) P^{-1} e + 2e^T P^{-1} \Delta$$

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If $P(t)$ sat. differential Riccati equation ($P(t_0) \in \mathcal{S}^n_{>0}$, $R \in \mathcal{S}^n_{>0}$)

$$\dot{P}(t) = P(t)A(t)^T + A(t)P(t) - P(t)C(t)^T QC(t)P(t) + R^{-1}$$

then

$$\dot{V}(e) = -e^T P^{-1}\left(PC^T QCP + R^{-1}\right)P^{-1} e + 2e^T P^{-1}\Delta$$

$\rightsquigarrow \dot{V}(e) \leq 0$ for e small enough

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Note that:

- For A, L, C constant also $P(t)$ is constant, i.e.,

$$\dot{P}(t) = 0 = PA^T + AP - PC^T QCP + R^{-1}$$

Extended Kalman Filter (Continuous Time) (3)

Consider (nonlinear) system with (nonlinear) output:

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)), \qquad x(0) \in \mathbb{R}^n \\ y(t) &= h(x(t)). \end{aligned}$$

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Error dynamics

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Lemma

Consider the error dynamics and let $R \in \mathcal{S}^n_{>0}$ *and* $Q \in \mathcal{S}^p_{>0}$ *be given. Additionally, for* $P(t_0) = P_0 \in \mathcal{S}^n_{>0}$, *assume that the solution* $P : \mathbb{R} \to \mathbb{R}^{p \times p}$ *of the Riccati differential equation exists for all* $t \geq t_0$ *and satisfies bounds for* $\alpha_1, \alpha_2 \in \mathbb{R}_{>0}$. *Then, for* $L(t) = P(t)C(t)^T Q$ *the origin is locally exponentially stable; i.e., there exist* $\delta, \lambda, M > 0$ *such that if* $|e(t_0)| \leq \delta$ *then for all* $t \geq t_0$,

$$|e(t)| \leq M |e(t_0)| \exp(\lambda(t - t_0)).$$

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$$|e(t)| \leq M|e(t_0)|\exp(\lambda(t - t_0)).$$

Note that:

- For nonlinear systems, only local exponential stability $\hat{x}(t) \to x(t)$ is satisfied
- Equations of the extended Kalman filter:

$$\begin{aligned} \dot{\hat{x}}(t) &= f(\hat{x}(t), u(t)) \\ &\quad + P(t) \left(\frac{\partial h}{\partial x}(\hat{x}(t))\right)^T Q(y(t) - h(\hat{x}(t))) \\ \dot{P}(t) &= P \left(\frac{\partial f}{\partial x}(\hat{x}(t), u(t))\right)^T + \left(\frac{\partial f}{\partial x}(\hat{x}(t), u(t))\right) P \\ &\quad - P \left(\frac{\partial h}{\partial x}(\hat{x}(t))\right)^T Q \left(\frac{\partial h}{\partial x}(\hat{x}(t))\right) P + R^{-1} \end{aligned}$$

- Initial conditions: $\hat{x}(t_0) = \hat{x}_0 \in \mathbb{R}^n$, $P(t_0) = P_0 \in \mathcal{S}^n_{>0}$
- For given $u : \mathbb{R}_{\geq t_0} \to \mathbb{R}^m$, $y : \mathbb{R}_{\geq t_0} \to \mathbb{R}^p$, the solution provides an approximation of $x(t)$ with guaranteed convergence $\hat{x}(t) \to x(t)$ for $t \to \infty$ if the assumptions of the Lemma are satisfied.
- The system of ODEs has to be solved in parallel. $P(t)$ is symmetric, i.e., the matrix equation can be written as an ODE of dimension $n(n+1)/2$.

Extended Kalman Filter (Continuous Time) (4)

Example (Inverted pendulum on a cart)

$$f(x,u)=\begin{bmatrix} x_3 \\ x_4 \\ \frac{-\bar{J}\bar{c}x_3-\bar{J}\sin(x_2)x_4^2-\bar{\gamma}\cos(x_2)x_4+g\cos(x_2)\sin(x_2)+\bar{J}u}{\bar{M}\bar{J}-\cos^2(x_2)} \\ \frac{-\bar{M}\bar{\gamma}x_4+\bar{M}g\sin(x_2)-\bar{c}\cos(x_2)x_3-\cos(x_2)\sin(x_2)x_4^2+\cos(x_2)u}{\bar{M}\bar{J}-\cos^2(x_2)} \end{bmatrix}$$

$$h(x)=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rightsquigarrow \frac{\partial h}{\partial x}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Partial derivatives of f:

$$\frac{\partial f}{\partial x_1}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}, \frac{\partial f}{\partial x_3}=\begin{bmatrix} 1 \\ 0 \\ \frac{\bar{c}\bar{J}}{\cos^2(x_2)-\bar{J}\bar{M}} \\ \frac{\bar{c}\cos(x_2)}{\cos^2(x_2)-\bar{J}\bar{M}} \end{bmatrix}, \frac{\partial f}{\partial x_4}=\begin{bmatrix} 0 \\ 1 \\ \frac{\bar{\gamma}\cos(x_2)+2\bar{J}x_4\sin(x_2)}{\cos^2(x_2)-\bar{J}\bar{M}} \\ \frac{\bar{\gamma}\bar{M}+2x_4\cos(x_2)\sin(x_2)}{\cos^2(x_2)-\bar{J}\bar{M}} \end{bmatrix}$$

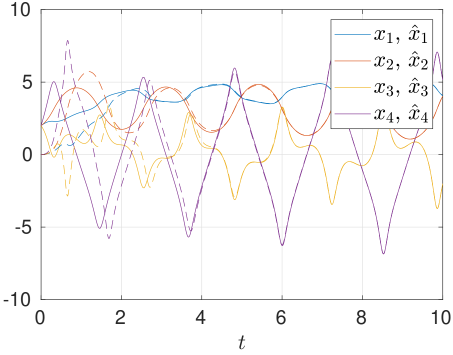
$$\frac{\partial f}{\partial x_2}=\cdots$$ (The expression is too long)

($\rightsquigarrow$ Use `syms` and symbolic differentiation `diff.m` in Matlab to obtain expressions)

$$A(t)=\frac{\partial f}{\partial x}=\begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} & \frac{\partial f}{\partial x_4} \end{bmatrix}$$

Selection of the input

$$u(\hat{x})=-\hat{x}_3(t)-\hat{x}_4(t).$$



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$$h(x)=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \rightsquigarrow \quad \frac{\partial h}{\partial x}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

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$$\frac{\partial f}{\partial x_2}=\cdots$$ (The expression is too long)

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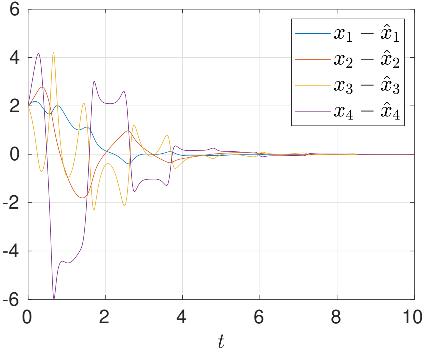
$$A(t)=\frac{\partial f}{\partial x}=\begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} & \frac{\partial f}{\partial x_4} \end{bmatrix}$$

Selection of the input

$$u(\hat{x})=-\hat{x}_3(t)-\hat{x}_4(t).$$



Extended Kalman Filter (Continuous Time) (4)

Example (Inverted pendulum on a cart)

$$f(x,u)=\begin{bmatrix} x_3 \\ x_4 \\ \frac{-\bar{J}\bar{c}x_3-\bar{J}\sin(x_2)x_4^2-\bar{\gamma}\cos(x_2)x_4+g\cos(x_2)\sin(x_2)+\bar{J}u}{\bar{M}\bar{J}-\cos^2(x_2)} \\ \frac{-\bar{M}\bar{\gamma}x_4+\bar{M}g\sin(x_2)-\bar{c}\cos(x_2)x_3-\cos(x_2)\sin(x_2)x_4^2+\cos(x_2)u}{\bar{M}\bar{J}-\cos^2(x_2)} \end{bmatrix}$$

$$h(x)=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \rightsquigarrow \quad \frac{\partial h}{\partial x}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Partial derivatives of f:

$$\frac{\partial f}{\partial x_1}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \frac{\partial f}{\partial x_3}=\begin{bmatrix} 1 \\ 0 \\ \frac{\bar{c}\bar{J}}{\cos^2(x_2)-\bar{J}\bar{M}} \\ \frac{\bar{c}\cos(x_2)}{\cos^2(x_2)-\bar{J}\bar{M}} \end{bmatrix}, \quad \frac{\partial f}{\partial x_4}=\begin{bmatrix} 0 \\ 1 \\ \frac{\bar{\gamma}\cos(x_2)+2\bar{J}x_4\sin(x_2)}{\cos^2(x_2)-\bar{J}\bar{M}} \\ \frac{\bar{\gamma}\bar{M}+2x_4\cos(x_2)\sin(x_2)}{\cos^2(x_2)-\bar{J}\bar{M}} \end{bmatrix}$$

$$\frac{\partial f}{\partial x_2}=\cdots$$ (The expression is too long)

($\rightsquigarrow$ Use `syms` and symbolic differentiation `diff.m` in Matlab to obtain expressions)

$$A(t)=\frac{\partial f}{\partial x}=\begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} & \frac{\partial f}{\partial x_4} \end{bmatrix}$$

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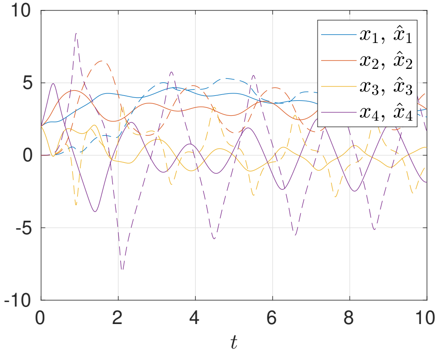

$\rightsquigarrow$ Only local convergence! (Here, different selection of Q.)

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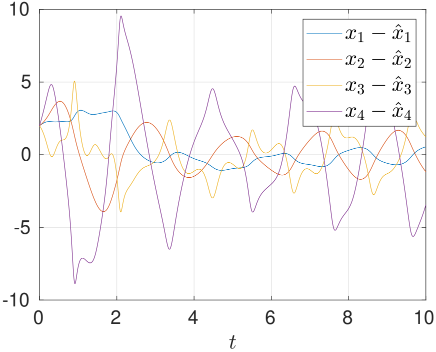

$\rightsquigarrow$ Only local convergence! (Here, different selection of Q.)

Section 2

Extended Kalman Filter (Discrete Time)

Extended Kalman Filter (Discrete Time)

Consider: Discrete time system

$$x(k+1) = f(k, x(k)) + g(k, x(k))v(k)$$
$$y(k) = h(k, x(k)) + w(k)$$

- state $x \in \mathbb{R}^n$; measured output $y \in \mathbb{R}^p$; unknown disturbances/noise $(v(k))_{k\in\mathbb{N}} \subset \mathbb{R}^q$; $(w(k))_{k\in\mathbb{N}} \subset \mathbb{R}^p$
- $f : \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}^n$, $g = [g_1, \ldots, g_q]$, $g_1, \ldots, g_q : \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}^n$, and $h : \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}^p$ are continuously differentiable by assumption.

Note that:

- $(u(k))_{k\in\mathbb{N}} \subset \mathbb{R}^m$ can be incorporated through the time dependence, i.e.,

$$f(k, x(k)) = \tilde{f}(x(k), u(k)), \;\; g(k, x(k)) = \tilde{g}(x(k), u(k)),$$
$$h(k, x(k)) = \tilde{h}(x(k), u(k)),$$

Extended Kalman Filter (Discrete Time)

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Assumptions: (zero mean Gaussian white noise)

$$\mathrm{E}\left[v(k)v(j)^T\right] = \begin{cases} Q^{-1}, & \text{if } k = j \\ 0, & \text{if } k \neq j \end{cases},$$

$$\mathrm{E}\left[w(k)w(j)^T\right] = \begin{cases} R^{-1}, & \text{if } k = j \\ 0, & \text{if } k \neq j \end{cases}$$

$\forall j, k \in \mathbb{N}$ and $Q \in \mathcal{S}^q_{>0}$, $R \in \mathcal{S}^p_{>0}$.

$$\mathrm{E}\left[v(k)w(j)^T\right] = 0, \; \mathrm{E}\left[v(k)x_0^T\right] = 0, \; \mathrm{E}\left[w(k)x_0^T\right] = 0,$$

$\forall j, k \in \mathbb{N}$ and for all initial conditions $x_0 \in \mathbb{R}^n$.

Extended Kalman Filter (Discrete Time)

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$\forall j, k \in \mathbb{N}$ and for all initial conditions $x_0 \in \mathbb{R}^n$.

Filter/Observer equations:

$$\hat{\chi}(k) = f(k-1, \hat{x}(k-1))$$
$$\hat{x}(k) = \hat{\chi}(k) + G_k\left(y(k) - h(k, \hat{\chi}(k))\right).$$

Task:

- How to define the Kalman gain matrix G_k?

Extended Kalman Filter (Discrete Time)

- Decompose x and y (stochastic/deterministic part)

$$\hat{x} = \hat{x}_d + \hat{x}_s, \qquad y = \hat{y}_d + \hat{y}_s$$

- Deterministic dynamics:

$$\begin{aligned} \hat{x}_d(k+1) &= f(k, \hat{x}_d(k)) \\ \hat{y}_d(k) &= h(k, \hat{x}_d(k)). \end{aligned}$$

- Stochastic dynamics:

$$\begin{aligned} \hat{x}_s(k+1) &= \hat{x}(k+1) - \hat{x}_d(k+1) \\ &= f(k, \hat{x}_d(k) + \hat{x}_s(k)) + g(k, \hat{x}(k))\hat{v}(k) - f(k, \hat{x}_d(k)) \end{aligned}$$

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- First order Taylor approximation of $f(k, \cdot)$ around $\hat{x}_d(k)$:

$$\hat{x}_s(k+1) \approx \frac{\partial f}{\partial x}(k, \hat{x}_d(k))\hat{x}_s(k) + g(k, \hat{x}(k))\hat{v}(k)$$

- Taylor approx. of $g(k, \cdot)$ around $\hat{x}_d(k)$ ($\hat{y}_s(k) = y(k) - \hat{y}_d(k)$):

$$\begin{aligned} \hat{y}_s(k) &\approx h(k, \hat{x}_d(k)) + \frac{\partial h}{\partial x}(k, \hat{x}_d(k))\hat{x}_s(k) + \hat{w}(k) - h(k, \hat{x}_d(k)) \\ &= \frac{\partial h}{\partial x}(k, \hat{x}_d(k))\hat{x}_s(k) + \hat{w}(k) \\ &= \frac{\partial h}{\partial x}(k, f(k-1, \hat{x}_d(k-1))\hat{x}_s(k) + \hat{w}(k) \end{aligned}$$

Extended Kalman Filter (Discrete Time)

- Decompose x and y (stochastic/deterministic part)

$$\hat{x} = \hat{x}_d + \hat{x}_s, \qquad y = \hat{y}_d + \hat{y}_s$$

- Deterministic dynamics:

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- With $\hat{x}_s(k)$ and $\hat{y}_s(k)$, we define A, $\bar{B}$, C:

$$\begin{aligned} A(k) &= \frac{\partial f}{\partial x}(k, \hat{x}(k)), \quad \bar{B}(k) = g(k, \hat{x}(k)) \\ C(k) &= \frac{\partial h}{\partial x}(k, \hat{\chi}(k)) \end{aligned}$$

$\leadsto$ Matrices define a linear time varying system

Extended Kalman Filter (Discrete Time)

- Decompose x and y (stochastic/deterministic part)

$$\hat{x} = \hat{x}_d + \hat{x}_s, \qquad y = \hat{y}_d + \hat{y}_s$$

- Deterministic dynamics:

$$\hat{x}_d(k+1) = f(k, \hat{x}_d(k))$$
$$\hat{y}_d(k) = h(k, \hat{x}_d(k)).$$

- Stochastic dynamics:

$$\begin{aligned}\hat{x}_s(k+1) &= \hat{x}(k+1) - \hat{x}_d(k+1)\\ &= f(k, \hat{x}_d(k) + \hat{x}_s(k)) + g(k, \hat{x}(k))\hat{v}(k) - f(k, \hat{x}_d(k))\end{aligned}$$

- First order Taylor approximation of $f(k, \cdot)$ around $\hat{x}_d(k)$:

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$$C(k) = \frac{\partial h}{\partial x}(k, \hat{\chi}(k))$$

$\rightsquigarrow$ Matrices define a linear time varying system

- We adapt the equations of the linear Kalman filter:

$$\begin{aligned}P_k^{k-1} =& \left[\frac{\partial f}{\partial x}(k-1, \hat{x}(k-1))\right] P_{k-1} \left[\frac{\partial f}{\partial x}(k-1, \hat{x}(k-1))\right]^T\\ &+ g(k-1, \hat{x}(k-1))Q^{-1} g(k-1, \hat{x}(k-1))^T\end{aligned}$$

$$\begin{aligned}G_k =& P_k k - 1 \left[\frac{\partial h}{\partial x}(k, \hat{\chi}(k))\right]^T\\ &\cdot \left(\left[\frac{\partial h}{\partial x}(k, \hat{\chi}(k))\right] P_k^{k-1} \left[\frac{\partial h}{\partial x}(k, \hat{\chi}(k))\right]^T + R^{-1}\right)^{-1}\end{aligned}$$

$$P_k = \left(I - G_k \left[\frac{\partial h}{\partial x}(k, \hat{\chi}(k))\right]\right) P_k^{k-1}$$

Extended Kalman Filter (3)

Input: Discrete time system with output, positive definite weight matrices $Q^{-1} = \mathrm{Var}(v(k))$, $R^{-1} = \mathrm{Var}(w(k))$, and initial estimates $\hat{x}(0) = \hat{x}_0$, $P_0 \in \mathcal{S}^n_{>0}$.

Output: Estimates $\hat{x}(k)$ and $\hat{\chi}(k)$ of the state $x(k)$ for $k \in \mathbb{N}$.

Algorithm: For $k \in \mathbb{N}$:

1. Compute $$\hat{\chi}(k) = f(k-1, \hat{x}(k-1))$$ and $$A(k-1) = \frac{\partial f}{\partial x}(k-1, \hat{x}(k-1)), \qquad \bar{B}(k-1) = g(k-1, \hat{x}(k-1)), \qquad C(k) = \frac{\partial h}{\partial x}(k, \hat{\chi}(k)).$$
2. Update the gain matrix $$P_k^{k-1} = A(k-1)P_{k-1}A(k-1)^T + \bar{B}(k-1)Q^{-1}\bar{B}(k-1)^T,$$ $$G_k = P_k^{k-1}C(k)^T\left[C(k)P_k^{k-1}C(k)^T + R^{-1}\right]^{-1},$$ $$P_k = \left[I - G_kC(k)\right]P_k^{k-1}.$$
3. Measure the output $y(k)$ and update the state estimate $$\hat{x}(k) = \hat{\chi}(k) + G_k\left(y(k) - h(k, \hat{\chi}(k))\right)$$ set $k = k+1$ and go to step 1.

Section 3

Moving Horizon Estimation

Moving Horizon Estimation

Recall:

- The dual to LQR is MME (Minimum energy estimator)
- We have discussed MPC as an extension of LQR
- The dual to MPC is MHE (Moving Horizon Estimation)

Consider:

$$x(k+1) = f(k, x(k), v(k))$$
$$y(k) = h(k, x(k)) + w(k)$$

- State $x \in \mathbb{X} \subset \mathbb{R}^n$; measured output $y \in \mathbb{R}^p$; $(v(k))_{k\in\mathbb{N}} \subset \mathbb{V} \subset \mathbb{R}^q$ and unknown disturbances/noise $(w(k))_{k\in\mathbb{N}} \subset \mathbb{W} \subset \mathbb{R}^p$
- Constraints: $\mathbb{X}, \mathbb{V}, \mathbb{W}$
- (Inputs $u(k)$ can be included as before)
- Goal: Based on measured data $y(k)$, find "optimal" $\hat{v}(k)$, $\hat{w}(k)$ such that

$$\hat{x}(k+1) = f(k, \hat{x}(k), \hat{v}(k)),$$
$$y(k) = h(k, \hat{x}(k)) + \hat{w}(k).$$

$\rightsquigarrow$ optimal state estimates $\hat{x}(k)$

- Define $\mathbb{D} = \mathbb{X} \times \mathbb{V} \times \mathbb{W}$

- At time $k \in N$, for given $y(i)$ for $i \in \mathbb{Z}_{[k-\bar{N},k-1]}$, define the set of feasible disturbance trajectories

$$\mathcal{V}_{\mathbb{D}}^{\bar{N}} = \left\{ v_{\bar{N}} : \mathbb{Z}_{[k-\bar{N},k-1]} \to \mathbb{R}^q \;\middle|\; \begin{array}{c} \hat{x}(i+1) = f(i, \hat{x}(i), v(i)) \\ y(i) = h(i, \hat{x}(i)) + w(i) \\ (\hat{x}(i+1), v(i), w(i)) \in \mathbb{D} \\ \forall\, i \in \mathbb{Z}_{[k-\bar{N},k-1]} \end{array} \right\}$$

(Note that $\mathcal{V}_{\mathbb{D}}^{\bar{N}} = \mathcal{V}_{\mathbb{D}}^{\bar{N}}(k, y_{\bar{N}})$ depends on k)

- *Cost function*: $\bar{J}_{\bar{N}} : \mathbb{R}^n \times \mathcal{U}_{\mathbb{D}}^{N} \to \mathbb{R} \cup \{\infty\}$,

$$\bar{J}_{\bar{N}}(\hat{x}(k-\bar{N}), v_{\bar{N}}(\cdot); y_{\bar{N}}(\cdot))$$
$$= F_{\bar{N}}(\bar{x}(k-\bar{N})) + \sum_{i=k-N}^{k-1} \ell(v(i), y(i) - h(i, \hat{x}(i)))$$

- For given $\hat{x}(k-\bar{N})$, $v(\cdot)$ and $y(\cdot)$, $\hat{x}(\cdot)$ and $w(\cdot)$ are implicitly defined through the dynamics.
- Costs with respect to disturbance $\ell : \mathbb{R}^q \times \mathbb{R}^p \to \mathbb{R}$;
- Costs with respect to estimate of the state $F_{\bar{N}} : \mathbb{R}^n \to \mathbb{R}$ ('terminal costs')

Moving Horizon Estimation (2)

Moving horizon optimization problem:

$$\bar{V}_{\bar{N}}(k, y_{\bar{N}}(\cdot)) = \min_{\substack{v_{\bar{N}}(\cdot) \in \mathcal{V}_{\mathbb{D}}^{\bar{N}} \\ \hat{x}(k-\bar{N}) \in \mathbb{X}}} \bar{J}_{\bar{N}}(\hat{x}(k-\bar{N}), v_{\bar{N}}(\cdot); y_{\bar{N}(\cdot)})$$

$$\text{subject to } \hat{x}(k+1) = f(k, \hat{x}(k), \hat{v}(k))$$

Note that:

- Optimal $\hat{v}_{\bar{N}}(\cdot)$ and optimal state estimate $\hat{x}(k-\bar{N})$ can be obtained from the solution
- Optimality is achieved w.r.t. a particular cost function.
- From $\hat{v}_{\bar{N}}(\cdot)$ and $\hat{x}(k-\bar{N})$ we obtain $\hat{x}(k)$
- The estimate $\hat{x}(k)$ can be used to design a state feedback $\mu(k) = u(\hat{x}(k))$ through MPC, for example.
- Similar to MPC, after shifting the horizon by going from k to $k+1$, the shifted optimization problem can be solved at the next time step to obtain $\hat{x}(k+1)$.

Introduction to Nonlinear Control

Stability, control design, and estimation

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Part IV:

Chapter 17: The Extended Kalman Filter

17.1 Extended Kalman Filter (Continuous Time)
17.2 Extended Kalman Filter (Discrete Time)
17.3 Moving Horizon Estimation

Australian National University